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Rearrangements of functions and partial differential equations for steady vortices

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Summary

Rearrangement variational principles are used to prove the existence of weak solutions of partial differential equations in connection with steady vortices of finite extent in an ideal fluid. The variational principles are based on that proposed by Benjamin [6] in which a functional related to the kinetic energy is maximised over the set of rearrangements of a fixed function.

The first two variational problems considered are examples in which a loss of compactness occurs. We first prove the existence of planar flows past an obstacle containing symmetric vortex pairs and approaching a uniform flow at infinity. The vorticity in one of the regions bounded by the line of symmetry is the maximiser of a functional relative to the set of rearrangements of a prescribed function that have bounded support. The functional is shown to attain a maximum for sufficiently small values of a positive parameter which corresponds to the speed of the flow at infinity.

The second problem we study concerns steady vortex rings in flows occupying the whole of $\mathbb{R}^3$. For all positive values of a parameter a functional is shown to attain a maximum relative to functions in the weak closure of the set of rearrangements of a prescribed function that have bounded support. The maximisers are rearrangements of curtailments of the prescribed function. In the special case when the prescribed function is constant the maximisers are rearrangements if the parameter is lower than a critical value and zero is the unique maximiser for values of the parameter greater than the critical value.

The final problem concerns periodic flows in a strip. We prove the existence of flows containing (disjoint) patches of vorticity with the vorticity in each patch being a rearrangement of a prescribed function.

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Contents

Chapter 1

Introduction

The theory of rearrangements and maximisation of functionals relative to sets of rearrangements has recently been developed by Burton. He has shown the existence of steady vortices in flows on both bounded and unbounded domains (planar and 3-dimensional) using a variational principle based on that proposed by Benjamin [6] in which a functional is maximised relative to the set of rearrangements of a prescribed function. In the case of unbounded domains, if the speed at infinity is prescribed to be sufficiently large then there is no maximiser of the functional relative to the set of rearrangements. However, there may exist a maximiser relative to the weak closure of the set of rearrangements. Douglas [15] has studied this problem in connection with vortex rings in an infinite pipe and showed that the maximiser is a rearrangement of a curtailment of the prescribed function.

Both Burton and Douglas used the technique of Steiner-symmetrisation to overcome the loss of compactness when working on unbounded domains. In the first problem we consider, that of planar flows past an obstacle, their methods are unavailable due to the lack of Steiner-symmetry of the domain. Symmetry techniques are used in the second problem we study, that of steady vortex rings in flows occupying the whole of $\mathbb{R}^3$. The final problem considered concerns periodic flows in a strip.

Before outlining the main results we give a brief summary of background theory relating to rearrangements and we also survey literature relevant to the thesis.

1.1 Rearrangements

Let $(\Sigma, \mathcal{M}, \mu)$ and $(\Sigma', \mathcal{M}', \mu')$ be positive measure spaces. Non-negative measurable functions $f : \Sigma \to \mathbb{R}$ and $g : \Sigma' \to \mathbb{R}$ are rearrangements if

$$\mu\{f^{-1}[\alpha, \infty)\} = \mu'\{g^{-1}[\alpha, \infty)\}$$

for all $\alpha > 0$. If $1 \leq p \leq \infty$ and $f \in L^p(\Sigma)$ then $g \in L^p(\Sigma')$ and $||f||_p = ||g||_p$.

The set of rearrangements of a non-negative function f_0 will be denoted by $R(f_0)$. For f_0 defined on the half-line Eydeland, Spruck and Turkington [19] characterised $R(f_0)$ as

$$R(f_0) = \{f | f \geq 0 \text{ , } f \text{ measurable, } \textstyle\int_0^\infty (f - \alpha)^+ = \int_0^\infty (f_0 - \alpha)^+ \text{ for all } \alpha > 0\}$$

where u^+ denotes the positive part of u. In general $R(f_0)$ is not a convex set.

Ryff [34] showed that for non-negative $f_0 \in L^1(I)$, where I is the unit interval, the weak closure of the set of rearrangements is convex and therefore equal to the closed convex hull of the set of rearrangements. Brown [8] proved that the result was true for any non-negative L^p function $(1 < p < \infty)$. Ryff also characterised the weak closure of the set of rearrangements of non-negative $f_0 \in L^1(I)$ as

$$\{f \in L^1(I) | \int_0^s f^\triangle \leq \int_0^s f_0^\triangle \text{ for } 0 < s < 1, \; f \geq 0 \text{ and } ||f||_1 = ||f_0||_1\}$$

where $f^\triangle$ denotes the decreasing function that is a rearrangement of f_0. The result for $f_0 \in L^p(I)$ $(1 < p < \infty)$ is easily deduced from the case $p = 1$.

For finite separable nonatomic measure spaces Burton [9] gave a direct proof of the convexity and weak sequential compactness of the weak closure of the set of rearrangements. Burton and Ryan [13] showed that the intersection of the weak closure of the set of rearrangements with a set of finitely many linear constraints is equal to the closed convex hull of the set of rearrangements intersected with the constraint set. They gave a characterisation of this set which coincides with that given by Ryff if there are no linear constraints.

For functions defined on unbounded domains two characterisations of the weak closure of the set of rearrangements were obtained by Douglas [16], one of these characterisations generalising that given by Ryff. We give some definitions before stating his results.

DEFINITION 1.1.1 *Let f, g be non-negative functions defined on the half-line and let $f^{\triangle}, g^{\triangle}$ denote the respective decreasing rearrangements on $(0, \infty)$. Then g is a curtailment of f at $l \in \overline{\mathbb{R}}$ if*

$$g = 1_{(0,l)} f^{\triangle}$$

and g is a rearrangement of a curtailment of f if $g^{\triangle}$ is a curtailment of f at some $l \in \overline{\mathbb{R}}$.

Let $1 < p < \infty$. Suppose $\Sigma \subset \mathbb{R}^n$ is open, unbounded and of infinite measure and μ is a non-zero, σ-finite, positive measure, absolutely continuous with respect to n-dimensional Lebesgue measure. Let $S : (0, \infty) \to \Sigma$ be a measure preserving transformation. The map $B : L^p(\Sigma) \to L^p(0, \infty)$ defined by

$$B(f) = f \circ S$$

is an isometry and for non-negative $f, g \in L^p(\Sigma)$ we have

$$g \in R(f) \text{ if and only if } B(g) \in R(B(f)).$$

DEFINITION 1.1.2 *Let $f \in L^p(\Sigma)$, $1 < p < \infty$, be non-negative. A non-negative function $g \in L^p(\Sigma)$ is a rearrangement of a curtailment of f if and only if $B(g)$ is a rearrangement of a curtailment of $B(f)$. The set of rearrangements of curtailments of f will be denoted by $RC(f)$.*

Let $f_0 \in L^p(\Sigma)$ be non-negative. The closure of $R(f_0)$ in the weak topology on L^p will be denoted by $\overline{R(f_0)}^w$.

Douglas [16] showed that

$$\overline{R(f_0)}^w = \{f \geq 0 | f \text{ measurable on } \Sigma, \textstyle\int_\Sigma (f - \alpha)^+ d\mu \leq \int_\Sigma (f_0 - \alpha)^+ d\mu \ \forall \alpha > 0\}.$$

Furthermore, $\overline{R(f_0)}^w = \overline{\text{conv } R(f_0)}$, the set of extreme points of $\overline{R(f_0)}^w$ is $RC(f_0)$ and $\overline{R(f_0)}^w$ is weakly sequentially compact.

1.1.1 Symmetrisation and rearrangement inequalities

DEFINITION 1.1.3 *Let $f \in L^p(\mathbb{R}^n)$ be non-negative. The spherically decreasing rearrangement, $\hat{f}$, of f is defined as*

$$\hat{f}(\mathbf{x}) = \sup\{t|\mu_n\{\mathbf{y}|f(\mathbf{y}) > t\} > \omega(n)|\mathbf{x}|^n\}$$

where $\omega(n)$ denotes the volume of the unit ball in $\mathbb{R}^n$.

For non-negative measurable functions f, g and h defined on $\mathbb{R}$, the inequality

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)g(x-y)h(y)dxdy \leq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{f}(x)\hat{g}(x-y)\hat{h}(y)dxdy$$

was proved by Riesz [33]. The analogous inequality for non-negative f, h defined on $\mathbb{R}^n$ and g a non-negative symmetrically strictly decreasing function on $\mathbb{R}$ is

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(\mathbf{x})g(|\mathbf{x}-\mathbf{y}|)h(\mathbf{y})d\mathbf{x}d\mathbf{y} \leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \hat{f}(\mathbf{x})g(|\mathbf{x}-\mathbf{y}|)\hat{h}(\mathbf{y})d\mathbf{x}d\mathbf{y}.$$

When the right-hand side is finite there is strict inequality unless f and h are translates of $\hat{f}$ and $\hat{h}$ respectively (see Lieb [28]).

Pólya and Szegö [32] showed that for $f \in W^{1,p}(\mathbb{R}^n)$, $1 \leq p < \infty$, the inequality

$$\int_{\mathbb{R}^n} |\nabla \hat{f}|^p d\mu_n \leq \int_{\mathbb{R}^n} |\nabla f|^p d\mu_n \tag{1.1.1}$$

holds. Brothers and Ziemer [7] considered the case when f has compact support and showed that if the distribution function $u(t) = \mu_n\{\hat{f} > t\}$ is absolutely continuous and equality holds in (1.1.1), then f is almost everywhere equal to a translate of $\hat{f}$.

1.1.2 Maximisation of linear and convex functionals

Burton [9] considers functions defined on finite positive measure spaces. Let $f_0 \in L^p(\Sigma)$ and $g_0 \in L^q(\Sigma)$ be non-negative where $1 \leq p \leq \infty$ and q denotes the conjugate exponent of p. Then for all rearrangements f of f_0 and g of g_0 on Σ we have

$$\int_{\Sigma} fg d\mu \leq \int_0^{\omega} f_0^{\Delta} g_0^{\Delta} \tag{1.1.2}$$

where $\omega = \mu(\Sigma)$. If there exists a rearrangement $\tilde{f}$ of f_0 with $\tilde{f} = \phi \circ g_0$ almost everywhere for some increasing function ϕ, then equality holds in (1.1.2) on taking $f = \tilde{f}$ and $g = g_0$.

Henceforth in this section $1 \leq p < \infty$ and as before $f_0 \in L^p(\Sigma)$, $g_0 \in L^q(\Sigma)$ are non-negative. For $w \in L^p(\Sigma)$ define

$$T(w) = \int_\Sigma w g_0.$$

Then if there exists a rearrangement $\tilde{f}$ of f_0 such that $\tilde{f} = \phi \circ g_0$ almost everywhere for some increasing function ϕ, $\tilde{f}$ is the unique maximiser of T relative to $\overline{R(f_0)}^w$. If the measure space under consideration is also nonatomic and separable, then there exists a rearrangement $\bar{f}$ of f_0 with $\int_\Sigma \bar{f} g_0 = \int_0^\omega f^\triangle g_0^\triangle$. Furthermore, if $\bar{f}$ is the unique maximiser of T relative to $R(f_0)$, then $\bar{f} = \phi \circ g_0$ almost everywhere for some increasing function ϕ.

These results are applied in [9, Theorem 7] to show that, in the case of finite separable nonatomic measure spaces, if $\Psi : L^p(\Sigma) \to \mathbb{R}$ is a weakly sequentially continuous convex functional then Ψ attains a maximum relative to $R(f_0)$. Furthermore, if Ψ is strictly convex, $\tilde{f}$ is a maximiser and $u \in \partial\Psi(\tilde{f})$ $(\subset L^q(\Sigma))$, then $\tilde{f} = \phi \circ u$ almost everywhere in Σ for some increasing function ϕ.

An immediate corollary is obtained by considering the functional

$$\psi(f) = \frac{1}{2}\int_\Sigma f K f d\mu - \int_\Sigma f v d\mu$$

for $f \in L^p(\Sigma)$ where $K : L^p(\Sigma) \to L^q(\Sigma)$ is a compact strictly positive symmetric linear operator and $v \in L^q(\Sigma)$. [9, Corollary 2] shows that ψ attains its supremum relative to $R(f_0)$ and if $\tilde{f}$ is any maximiser then $\tilde{f} = \phi \circ (K\tilde{f} - v)$ almost everywhere for some increasing function ϕ.

1.2 Existence theorems for steady vortices

Let (r, θ, z) denote cylindrical coordinates in $\mathbb{R}^3$. Let $\psi(r, z)$ denote the stream function for the flow of an ideal fluid. The velocity is given by

$$\mathbf{v} = (-\frac{1}{r}\psi_z, 0, \frac{1}{r}\psi_r)$$

where the subscripts denote partial derivatives, and the vorticity ω is given by

$$\operatorname{curl} \mathbf{v} = (0, \omega, 0).$$

Defining

$$\mathcal{L}\psi = -\frac{1}{r}\left(\frac{1}{r}\psi_r\right)_r - \frac{1}{r^2}\psi_{zz}$$

we obtain $\omega = r\mathcal{L}\psi$. The region where $\omega \neq 0$ is called the vortex core.

The approach of Benjamin [6] was to seek a steady flow for which ω/r is a rearrangement of a prescribed function, and for which a value is prescribed for either the speed at infinity, or the impulse which is given for a fluid of unit density by

$$P(\omega) = \int_{\mathbb{R}^3} r\omega.$$

Both P and the measures of the sets $\{\omega/r \geq \alpha\}$ are preserved in all axisymmetric motions of an ideal fluid in $\mathbb{R}^3$. If $\mathcal{L}\psi = \phi \circ \psi$ for some function ϕ then ψ is the stream function for a steady flow.

Burton showed the existence of steady vortex rings in a bounded axisymmetric domain with C^2 boundary [9, Section 4]. For any real λ there exists a function ψ satisfying

$$\mathcal{L}\psi = \phi \circ (\psi - \lambda r^2/2) \tag{1.2.3}$$

almost everywhere for some increasing function ϕ unknown *a priori* with $\mathcal{L}\psi$ a rearrangement of a given function. The existence of flows with prescribed impulse is also considered and if the impulse is prescribed to be positive and satisfy a certain feasibility condition, then there exists a function satisfying (1.2.3) for some real λ and some increasing function ϕ. In each case the function $\psi - \lambda r^2/2$ is the Stokes stream function for the flow.

Flows in an infinite pipe of circular cross section have been considered by Burton [10] and Douglas [16]. The cylindrical symmetry allowed them to reduce the problem to one for functions defined on a strip endowed with an appropriate measure.

Let $f_0 \in L^p$ $(p > 5)$ be a non-zero non-negative function vanishing outside a set of finite measure. Burton considered the maximisation of a functional relative to the set of rearrangements of f_0 that have bounded support. The functional takes the form $E - \lambda I$ where E and I are related to the kinetic energy and impulse of the flow respectively and λ is a positive parameter. If λ is sufficiently small then the supremum of the functional is attained and there exists a function ψ satisfying (1.2.3) almost everywhere for some increasing function ϕ with $\mathcal{L}\psi$ a rearrangement of f_0.

Douglas proved that for any positive λ the functional attains a maximum relative to the weak closure of the set of rearrangements of a non-zero non-negative function $f_0 \in L^1 \cap L^p$ $(5/2 < p < \infty)$. The maximisers are rearrangements of curtailments of f_0 and give rise to weak solutions of (1.2.3).

The first general existence theorem for vortex rings was established by Fraenkel and Berger [21]. They showed that given the stream velocity $\lambda > 0$, the flux constant $\gamma \geq 0$, the kinetic energy of the vortex motion and a non-decreasing Hölder continuous vorticity function $\phi : \mathbb{R} \to [0, \infty)$, there exists a function u that satisfies

$$\begin{aligned} \mathcal{L}u &= -k\phi(u - \lambda r^2/2 - \gamma) \text{ pointwise in } \Pi, && (1.2.4) \\ u(0,z) &= 0 && (1.2.5) \\ u &\to 0 \text{ and } |\nabla u| \to 0 \text{ as } r^2 + z^2 \to \infty && (1.2.6) \end{aligned}$$

where k is the vortex strength parameter which is unknown *a priori*. If the vorticity function is allowed to have a simple discontinuity at 0 then there is a function u that satisfies (1.2.4) almost everywhere. The variational principle involves maximisation of a functional relative to those functions having prescribed kinetic energy.

The only known explicit solution of (1.2.4), together with the boundary conditions (1.2.5) and (1.2.6), was discovered by Hill [25] and corresponds to the case $\gamma = 0$ and $\phi = \phi_h$ where ϕ_h is the Heaviside function given by

$$\phi_h(t) = \begin{cases} 1 & \text{if } t > 0, \\ 0 & \text{if } t \leq 0. \end{cases}$$

Amick and Fraenkel [3] proved that the weak solutions obtained in [21] (with $\gamma = 0$ and $\phi = \phi_h$) are Hill's solution, modulo translation in the z-direction. Indeed, any weak solution of Hill's problem (defined in 3.5.1) is Hill's solution modulo translation in the z-direction.

Norbury [30] used a contraction principle to show that given k, λ and sufficiently small $\gamma > 0$, there are solutions that are close to Hill's vortex. Amick and Fraenkel [4] established that any solution in a sufficiently small neighbourhood of Hill's vortex (for prescribed k and λ) coincides with Norbury's solution. The local branch is actually a subset of the global branch emanating from Hill's vortex found by Amick and Turner [5].

Friedman and Turkington [22] considered a variational problem in which the impulse is prescribed. They proved the existence of vortex rings in which ω/r is equal to a constant, k. Estimates for the diameter of the vortex core are used to show that, as $k \to \infty$, the vortex core (in the (r,z)-plane) is asymptotically a disc centred at a point on the r-axis.

In two dimensional ideal fluid flows the stream function $\psi(x_1, x_2)$ gives rise to a velocity

$$\mathbf{v} = (\psi_{x_2}, -\psi_{x_1})$$

and the vorticity ω is given by

$$-\Delta\psi = \omega.$$

Norbury [31] proved the existence of vortex pairs in $\mathbb{R}^2$ using a method analogous to that used by Fraenkel and Berger [21] for vortex rings, the vorticity function being prescribed.

Burton [11] used a rearrangement variational principle in which the rearrangement class of the vorticity field and the speed at infinity are prescribed. Let $\Pi = \{(x_1, x_2) \in \mathbb{R}^2 | x_2 > 0\}$. Let $f_0 \in L^p(\Pi)$ $(p > 2)$ have bounded support. For $v \in L^p(\Pi)$ having bounded support and $\lambda > 0$ the functional

$$\frac{1}{2}\int_\Pi vT_0v - \lambda\int_\Pi x_2 v,$$

where T_0 is an inverse operator to $-\Delta$, is shown to attain a maximum relative to the set of rearrangements of f_0 supported in a fixed rectangle. For λ sufficiently small the maximisers for sufficiently large rectangles are supported in a fixed rectangle and as a consequence can be shown to give a solution valid in Π.

Turkington [36, 37] proved the existence of vortex pairs in flows occupying the whole of $\mathbb{R}^2$ and also flows in $\Omega^* = \mathbb{R}^2 \backslash D$ where D is a bounded simply connected domain, symmetric in the x_1-axis and having smooth boundary. Let

$\Omega = \{\mathbf{x} \in \Omega^* | x_2 > 0\}$. For $\mathbf{b} = (b_1, b_2)$ let

$$\Omega^{\mathbf{b}} = \{\mathbf{x} \in \Omega | |x_1| < b_1, 0 < x_2 < b_2\}.$$

For $\lambda > 0$, Turkington considered maximisation of the functional

$$\frac{1}{2}\int_\Omega wTw - \lambda \int_\Omega \eta w,$$

where T is an inverse operator to $-\Delta$ and η is the stream function for an irrotational flow with the same boundary conditions but approaching x_2 at infinity, relative to those non-negative functions $w \in L^\infty(\Omega^{\mathbf{b}})$ satisfying the constraints $||w||_1 = 1$ and $||w||_\infty \leq \theta$. He showed that if $b_1, b_2 > M(\lambda, D)$ and $\theta > \theta(\boldsymbol{b}, \lambda, D)$ then the support of any maximiser, $w^{\mathbf{b}}_{\lambda,\theta}$, of the functional relative to functions vanishing outside $\Omega^{\mathbf{b}}$ is bounded away from $x_1 = \pm b_1$ and $x_2 = b_2$ and $w^{\mathbf{b}}_{\lambda,\theta}$ is equal to a constant, θ, on a set of measure $1/\theta$. Also $u^{\mathbf{b}}_{\lambda,\theta}$ provides the vorticity of a solution of the Euler equations on Ω but, because of the dependence of θ on $\mathbf{b}$, it does not immediately follow that $u^{\mathbf{b}}_{\lambda,\theta}$ maximises the functional relative to functions supported in Ω. Asymptotic properties of the maximisers are established, in particular $u^{\mathbf{b}}_{\lambda,\theta} \to \delta_X$ in the sense of distributions as $\theta \to \infty$ where X is the minimiser of the Routh function (which is $h(\mathbf{x},\mathbf{x})/2 + \lambda\eta(\mathbf{x})$ where $h(\mathbf{x},\mathbf{y})$ is the harmonic part of the Green's function) corresponding to the region.

For $\lambda > 0$ let $c = 1/\lambda$. Let $\omega_0 \in L^p(\Omega^*)$ $(2 < p < \infty)$ and define $\zeta_c(\mathbf{x}) = c^2\omega_0(c\mathbf{x})$.

Elcrat and Miller [18] have shown that if $c > c(\omega_0, D)$ is sufficiently large there exists a steady flow with prescribed circulation around the obstacle, stream function x_2 at infinity and vorticity a rearrangement of ζ_c. They consider maximisation of a functional (which is independent of λ) relative to rearrangements of ζ_c supported on a compact subdomain of Ω^* which contains a minimiser of the Routh function as an interior point. The dependence of estimates for the diameter of the support of the maximisers on the size of the domain is not examined and the existence of flows with vorticity a rearrangement of ω_0 follows only if the obstacle is sufficiently large depending on ω_0.

Keady [27] has studied periodic flows defined in a strip. The problem is reduced to one on a rectangle with mixed boundary conditions. He considers maximisation of a functional relative to a class of sets that have fixed centroid and fixed area. It is conjectured that a maximising set whose closure is bounded away from the edges of the rectangle corresponds to the vortex core of a flow. However, the existence of flows with vortex core bounded away from the edges of the rectangle is not established. Under the assumption that such flows exist, asymptotic results concerning the diameter, capacity and centroid (which is prescribed) of the vortex core are obtained.

1.3 Summary of main results

In Chapter 2 we combine the methods of Burton [11] and Turkington [36, 37] to prove an existence theorem for a steady planar flow of an ideal fluid past an obstacle, containing symmetric vortex pairs and approaching a uniform flow at infinity. The vorticity in one of the regions bounded by the line of symmetry is a rearrangement of a prescribed function and the vortex core is bounded.

The method used is based on the observation that by scaling the functional and the prescribed function appropriately we can equivalently consider the maximisation of a functional relative to the set of rearrangements of a function which vanishes outside a set whose measure tends to 0 as $\lambda \to 0$.

We first consider the case when there is no obstacle. An alternative proof of [11, Theorem 16(i)] is given which uses local estimates of the diameter of the support of the maximisers that are similar to those derived by Turkington.

In the case of no obstacle the estimates only use Steiner-symmetry of the maximisers to ensure the maximisers are centred appropriately. When there is an obstacle we are still able to estimate the diameter of the support of the maximisers and the asymptotic results for flows without an obstacle are used to show that the maximisers actually have bounded support. We determine asymptotic properties of the maximisers analogous to those in [36, Section 4].

In Chapter 3 we prove the existence of steady vortex rings in flows occupying the whole of $\mathbb{R}^3$. For all $\lambda > 0$ (which corresponds to the speed that the flow approaches at infinity), a functional of the form $E - \lambda I$, where E and I are related to the kinetic energy and impulse of the flow respectively, is shown to attain a maximum relative to functions that are in the weak closure of the set of rearrangements of a prescribed function and that also have bounded support.

The maximisers give rise to weak solutions of the partial differential equation corresponding to an axisymmetric steady flow. Letting ω and r denote the vorticity and the distance from the axis of symmetry respectively, the maximisers represent the quantity ω/r, and each maximiser is a rearrangement of a curtailment of the prescribed function. The vortex core is a bounded set.

The method used is to show that there exist maximising sequences for the functional supported on a strip (because of the cylindrical symmetry we work on a half-plane) whose width depends on the prescribed function and the parameter. We then apply a result of Lions [29] regarding compact embeddings of function spaces with Steiner-symmetric elements to show that this maximum is attained.

In the special case where the prescribed function is constant in its support we use the uniqueness results of Amick and Fraenkel [3] to prove that for all values of λ below a critical value the maximisers are rearrangements whereas for all values of λ greater than the critical value the unique maximiser is zero. When λ is equal to the critical value a maximiser may be either a rearrangement or zero.

In Chapter 4 we prove the existence of periodic flows in a strip. The vorticity in each rectangle is a rearrangement of a prescribed function and the vortex core avoids the boundary of the rectangle. The method used is similar to that for the existence of vortex pairs in the plane. For all values of a positive parameter which represents the average velocity, a functional is shown to attain a maximum relative to the set of rearrangements of a prescribed non-negative function. If the parameter is sufficiently small and the dimensions of the rectangle are sufficiently large depending on the parameter, then the vortex core is bounded away from the edges of the rectangle.

Chapter 2

Vortex pairs on a planar domain with an obstacle

2.1 Introduction

In [11] Burton proves the existence of steady planar flows containing steady symmetric vortex pairs, the vorticity in one of the half-planes bounded by the line of symmetry being a rearrangement of a prescribed function. Turkington [36, 37] has shown the existence of steady flows past an obstacle in which the vorticity is constant and inversely proportional to the area of the vortex core in the domains into which the line of symmetry divides the flow.

We combine the methods of these papers to prove an existence theorem for a steady planar flow of an ideal fluid past an obstacle, containing symmetric vortex pairs and approaching a uniform flow at infinity. The vorticity in one of the regions bounded by the line of symmetry is a rearrangement of a prescribed function. The variational principle used is the same as in [11] which is an adaptation of a theory proposed by Benjamin [6] for vortex rings in three dimensions.

Specifically, let D denote the closure of a bounded simply-connected domain, symmetric in the x_1-axis with boundary of class C^2. Let $\Omega^* = \mathbb{R}^2 \backslash D$ and $\Omega = \{(x_1, x_2) \in \Omega^* | x_2 > 0\}$. The stream function $\psi : \Omega \to \mathbb{R}$ for the required flow is such that $\psi(x_1, x_2)$ is odd in x_2 and $\psi \sim -\lambda x_2$ at infinity. The vorticity is given by $-\Delta\psi$ and is zero outside a pair of bounded regions symmetric about the x_1 axis and avoiding the axis. The vorticity in the region $x_2 > 0$ is positive and is a rearrangement of a prescribed function ω_0 having bounded support. In each of the regions $\pm x_2 > 0$ the equation

$$-\Delta\psi = \phi^{\pm}(\psi)$$

is satisfied where $\phi^{\pm}$ are increasing functions, but $-\Delta\psi$ is not a function of ψ throughout Ω^*.

2.2 Description of the method

DEFINITION 2.2.1 *We shall say a non-negative measurable function $f : \mathbb{R}^2 \to \mathbb{R}$ is Steiner-symmetric if*

$$0 \le x_1 \le x_1' \Rightarrow f(-x_1, x_2) = f(x_1, x_2) \ge f(x_1', x_2) \ge 0$$

for almost every x_2. The Steiner-symmetrisation, $f^{\natural}$, of f is the essentially unique rearrangement of f that is Steiner-symmetric with

$$\mu_1\{t | f(t, x_2) \ge \alpha\} = \mu_1\{t | f^{\natural}(t, x_2) \ge \alpha\}$$

for almost every x_2 and every $\alpha > 0$.

Let $2 < p < \infty$. Let $\omega_0 \in L^p(\Omega)$ be non-negative and vanish outside a set of finite measure. Let $\mathcal{F}$ be the set of rearrangements of ω_0 on Ω having bounded support.

For $\lambda > 0$ and $v \in L^p(\Omega)$ having bounded support define

$$\Psi_\lambda(v) = \frac{1}{2}\int_\Omega vTv - \lambda \int_\Omega \eta v$$

where the operator T is inverse to $-\Delta$ with homogenous Dirichlet boundary conditions on Ω and approaching zero at infinity, and η is the stream function for an irrotational flow with the same boundary conditions as T but approaching x_2 at infinity.

Let $c = 1/\lambda$. Define

$$\Omega_c = \{\mathbf{x} \in \Pi | c\mathbf{x} \in \Omega\}.$$

Let $\zeta_c(\mathbf{x}) = c^2\omega_0(c\mathbf{x})$ and let $\mathcal{F}_c$ denote the set of rearrangements of ζ_c on Ω_c having bounded support. For $v \in L^p(\Omega_c)$ having bounded support define

$$\tilde{\Psi}_\lambda(v) = \frac{1}{2}\int_{\Omega_c} vT_c v - \int_{\Omega_c} \eta_c v$$

where the operator T_c is inverse to $-\Delta$ with homogenous Dirichlet boundary conditions on Ω_c and approaching zero at infinity, and η_c is the stream function for an irrotational flow with the same boundary conditions as T_c but approaching x_2 at infinity.

For $v \in L^p(\Omega)$ having bounded support and $\mathbf{x} \in \Omega_c$ define $v_c(\mathbf{x}) = c^2 v(c\mathbf{x})$. Then

$$\Psi_\lambda(v) = \tilde{\Psi}_\lambda(v_c).$$

For $\xi > 0$ let $\Omega_c(\xi) = \Omega_c \cap B_\xi(0)$ and let $\mathcal{F}_c$ denote the set of functions in $\mathcal{F}_c$ that vanish outside $\Omega_c(\xi)$. Then if $c\xi$ is sufficiently large, $\tilde{\Psi}_\lambda$ attains a maximum relative to $\mathcal{F}_c(\xi)$ and, letting $\tilde{\zeta}_{c,\xi}$ be a maximiser, we have

$$\tilde{\zeta}_{c,\xi} = \phi \circ (T_c\tilde{\zeta}_{c,\xi} - \eta_c)$$

almost everywhere in $\Omega_c(\xi)$ for some increasing function ϕ.

Let $A_{c,\xi} = \{\tilde{\zeta}_{c,\xi} > 0\}$. We use local estimates for the diameter of $A_{c,\xi}$ to show that for c and ξ sufficiently large $A_{c,\xi}$ is contained in a ball of fixed radius centred on the x_1-axis. We then use results from the case when there is no obstacle to prove the existence of ξ_0, C_0 such that $A_{c,\xi} \subset \Omega_c(\xi_0)$ except for a set of measure zero for all $\xi \geq \xi_0$, $c \geq C_0$. Hence $\tilde{\zeta}_{c,\xi_0}$ maximises $\tilde{\Psi}_\lambda$ relative to $\mathcal{F}_c$ and it follows that Ψ_λ attains a maximum relative to $\mathcal{F}$. If $\omega_{\lambda,\xi_0/\lambda}$ maximises Ψ_λ relative to $\mathcal{F}$ then it is shown that $\omega_{\lambda,\xi_0/\lambda} = \phi \circ (T\omega_{\lambda,\xi_0/\lambda} - \lambda\eta)$ almost everywhere in Ω for some increasing function ϕ.

As mentioned above, in order to show that Ψ_λ attains a maximum relative to $\mathcal{F}$ we require results from the case when there is no obstacle. In this case the explicit formula for the Green's function and the existence of Steiner-symmetric maximisers enables us to obtain more precise asymptotic results. The results for flows with no obstacle are given in 2.4 and those for flows past an obstacle in 2.5.

2.3 Preliminaries

In this section we define, for $v \in L^p(\Omega)$ with bounded support, an operator, T, which is inverse to $-\Delta$ with homogenous Dirichlet boundary conditions and approaching zero at infinity. We show that for a bounded open subset U of Ω, $T : L^p(U) \to W^{2,p}(U)$ is bounded for $2 < p < \infty$. The operator T is an integral operator with kernel the Green's function for $-\Delta$ with the same boundary conditions.

Burton [11] shows that an operator, T_0, with analogous properties may be defined for functions in $L^p(\Pi)$ with bounded support. The explicit formula for the Green's function for Π enables the calculation of asymptotic estimates for $T_0 v$ and $\nabla T_0 v$. For non-negative v the maximum principle yields $0 \leq Tv \leq T_0 v$ from which we obtain asymptotic estimates for Tv and ∇Tv.

2.3.1 Notation and definitions

Let

$$\Pi = \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 | x_2 > 0\}$$

and

$$\Theta = \{\mathbf{x} \in \Pi | |\mathbf{x}| > 1\}.$$

Let $D \subseteq \{\mathbf{x} \in \mathbb{R}^2 | |\mathbf{x}| \leq 1\}$ be the closure of a bounded, simply-connected domain, symmetric in the x_1 axis, such that ∂D is of class C^2 and define $\Omega^* = \mathbb{R}^2 \backslash D$. Let $\Omega = \Pi \backslash D$. For $\xi > 0$ define $\Pi(\xi) = \Pi \cap B_\xi(0)$ and define $\Omega(\xi)$, $\Omega^*(\xi)$ and $\Theta(\xi)$ similarly.

Let $g_0(\mathbf{x},\mathbf{y})$, $g(\mathbf{x},\mathbf{y})$ and $g_1(\mathbf{x},\mathbf{y})$ be the Green's functions for $-\Delta$ with homogenous Dirichlet boundary conditions on Π, Ω and Θ respectively and zero at infinity (for existence of the Green's functions see [14, Chapter 2, Section 4, Proposition 12]). We follow Turkington [36, 37] in defining the harmonic function

$$h(\mathbf{x},\mathbf{y}) = \frac{1}{2\pi} \log \frac{1}{|\mathbf{x}-\mathbf{y}|} - g(\mathbf{x},\mathbf{y})$$

and similarly let

$$h_i(\mathbf{x},\mathbf{y}) = \frac{1}{2\pi} \log \frac{1}{|\mathbf{x}-\mathbf{y}|} - g_i(\mathbf{x},\mathbf{y}), \qquad i = 0, 1$$

denote the corresponding harmonic functions for Π and Θ respectively. Then

$$\begin{aligned} g_0(\mathbf{x},\mathbf{y}) &= \frac{1}{2\pi}\log\left(\frac{|\mathbf{x}-\bar{\mathbf{y}}|}{|\mathbf{x}-\mathbf{y}|}\right) = \frac{1}{4\pi}\log\left(1+\frac{4x_2y_2}{|\mathbf{x}-\mathbf{y}|^2}\right), \ \forall \mathbf{x},\mathbf{y}\in\Pi \\ g_1(\mathbf{x},\mathbf{y}) &= g_0(\mathbf{x},\mathbf{y}) - \frac{1}{4\pi}\log\left(1+\frac{4x_2y_2}{|\mathbf{y}|^2|\mathbf{x}-|\mathbf{y}|^{-2}\mathbf{y}|^2}\right), \ \forall \mathbf{x},\mathbf{y}\in\Theta \end{aligned}$$

where $^-$ denotes reflection in the x_1 axis.

For a measurable function v on Π and $\mathbf{x}\in\mathbb{R}^2$ define

$$T_0v(\mathbf{x}) = \int_\Pi g_0(\mathbf{x},\mathbf{y})v(\mathbf{y})d\mathbf{y}$$

and similarly, for a measurable function v on Ω and $\mathbf{x}\in\Omega$, define

$$Tv(\mathbf{x}) = \int_\Omega g(\mathbf{x},\mathbf{y})v(\mathbf{y})d\mathbf{y}$$

whenever these integrals exist.

Throughout this chapter we shall use $M(*,...,*)$ to denote a constant depending on the quantities in parentheses. For an open subset U of $\mathbb{R}^2$ we shall denote by $|\cdot|_{k,\nu;U}$, $|\cdot|_{k;U}$, $||\cdot||_{k,p;U}$ and $||\cdot||_{p;U}$ the norms on $C^{k,\nu}(\overline{U})$, $C^k(\overline{U})$, $W^{k,p}(U)$ and $L^p(U)$ respectively.

2.3.2 Properties of T_0 and T

We state some results from [11].

LEMMA 2.3.1 (i) *Let $0<a<\infty$ and $1<p<\infty$. Then there are positive constants $A,B>0$ such that if $v\in L^p(\Pi)$ and v vanishes outside a set of area πa^2, then for $\mathbf{x}\in\mathbb{R}^2$ with $|x_2|\geq a$ we have*

$$|T_0v(\mathbf{x})| \leq (A+B\log|x_2|)||v||_p.$$

(ii) *Let $1<p<\infty$ and let $v\in L^p(\Pi)$ have bounded support. Then $T_0v\in W^{2,p}_{loc}(\mathbb{R}^2)$ and $-\Delta T_0 v = v_0$ almost everywhere in $\mathbb{R}^2$, where v_0 is the extension of v that is odd in x_2.*

(iii) *Let* $2 < p < \infty$ *and* $0 < a < \infty$. *Then for any* $v \in L^p(\Pi)$ *vanishing outside a set of area* πa^2, *we have* $T_0 v \in C^1(\mathbb{R}^2)$ *and*

$$|T_0 v(\mathbf{x})| \leq N|x_2|||v||_p$$

for all $\mathbf{x} \in \mathbb{R}^2$, *where* N *is a constant depending only on* a *and* p.

(iv) *Let* $2 < p < \infty$, *let* $1 \leq q < \infty$ *and let* U *be a bounded open subset of* Π. *Then* $T_0 : L^p(U) \to L^q(U)$ *is compact, in the sense that if* v_n *is a sequence of functions, bounded in* $L^p(\Pi)$ *and vanishing outside* U, *then the sequence* $T_0 v_n|_U$ *has a subsequence converging in the* q*-norm.*

(v) *Let* $v \in L^1(\Pi)$ *be non-negative and have bounded support. Then*

$$\int_\Pi v T_0 v \leq \int_\Pi v^\sharp T_0 v^\sharp,$$

and if v *is Steiner-symmetric then* $T_0 v$ *is Steiner-symmetric.*

(vi) *Let* $2 < p < \infty$ *and let* $v \in L^p(\Pi)$ *have bounded support. Then* $\nabla T_0 v(\mathbf{x}) = O(|\mathbf{x}|^{-2})$, $T_0 v(\mathbf{x}) = O(|\mathbf{x}|^{-1})$ *as* $|\mathbf{x}| \to \infty$, *and*

$$\int_\Pi |\nabla T_0 v|^2 = \int_\Pi v T_0 v < \infty.$$

We prove some analogous results for Ω.

LEMMA 2.3.2 *Let* $2 < p < \infty$ *and* $1 \leq q < \infty$. *Let* $v \in L^p(U)$ *where* U *is a bounded subset of* Ω. *Then* $T : L^p(U) \to L^q(U)$ *is compact,*

$$\begin{aligned} Tv &\in W^{2,p}(U), \\ -\Delta Tv &= v \text{ almost everywhere in } \Omega, \\ \text{and } Tv &= 0 \text{ on } \partial\Omega \text{ in the weak sense.} \end{aligned}$$

If v *is non-negative then*

$$0 \leq Tv \leq T_0 v \text{ almost everywhere in } \Omega.$$

(Note that T *is compact in the sense that if* v_n *is a sequence of functions, bounded in* $L^p(\Omega)$ *and vanishing outside* U, *then the sequence* $Tv_n|_U$ *has a subsequence*

converging in the q-norm)

Proof Let $n_1 \in \mathbb{N}$ be such that $U \subset \Omega(n_1)$. For $n \geq n_1$ define $T_n v$ as the unique minimiser over $H_0^1(\Omega(n))$ of the functional

$$\Phi_n(u) = \frac{1}{2}\int_{\Omega(n)} |\nabla u|^2 - \int_{\Omega(n)} uv.$$

Then

$$\begin{aligned} -\Delta T_n v &= v \quad \text{on } \Omega(n) \\ T_n v &= 0 \quad \text{on } \partial\Omega(n) \text{ in the weak sense.} \end{aligned} \tag{2.3.1}$$

and, in particular,

$$T_n v(\mathbf{x}) = \int_{\Omega(n)} g_n(\mathbf{x}, \mathbf{y}) v(\mathbf{y}) d\mathbf{y}$$

almost everywhere in $\Omega(n)$ where $g_n(\mathbf{x}, \mathbf{y})$ denotes the Green's function for $-\Delta$ with zero Dirichlet boundary conditions on $\Omega(n)$ (see for example [14, Chapter 2, Section 7, Corollary 3]).

Note that if v is non-negative then by the generalised weak maximum principle [23, Theorem 8.1]

$$0 \leq T_m v(\mathbf{x}) \leq T_n v(\mathbf{x}) \leq T_0 v(\mathbf{x})$$

for almost every $\mathbf{x} \in \Omega(m)$ and every $n \geq m \geq n_1$.

Let $\tilde{T}_n v \in H_0^1(\Omega^*(n))$ be the extension of $T_n v$ as an odd function of x_2. Fix $n_0 > n_1 + 1$. Then $\tilde{T}_n v \in W^{1,2}(\Omega^*(n_0))$ for all $n \geq n_0$ and $-\Delta \tilde{T}_n v = \tilde{v}$ in the weak sense where $\tilde{v}$ is the extension of v as an odd function of x_2.

Let $\phi \in C^\infty(\overline{\Omega^*(n_0)})$ be such that $\phi(\mathbf{x}) = 1$ for all $\mathbf{x} \in \Omega^*(n_1)$ and $\phi(\mathbf{x}) = 0$ for all $\mathbf{x} \in \Omega^*(n_0) \backslash \Omega^*(n_1 + 1)$. Then $\phi \tilde{T}_n v \in W_0^{1,2}(\Omega^*(n_0))$ for all $n \geq n_0$.

By Lemma 2.3.1 (iii) we have

$$\begin{aligned} \frac{1}{2}\int_{\Omega(n_0)} |\nabla T_n v|^2 \leq \frac{1}{2}\int_{\Omega(n)} |\nabla T_n v|^2 &\leq \int_{\Omega(n)} v T_n v \\ &\leq N ||v||_p \int_{\Omega(n_0)} x_2 |v| \\ &\leq M(n_0). \end{aligned} \tag{2.3.2}$$

Also

$$-\Delta(\phi\tilde{T}_n v) = (-\Delta\phi)\tilde{T}_n v - 2(\nabla\phi).(\nabla\tilde{T}_n v) + \phi\tilde{v}$$

in the weak sense and by (2.3.2) $||-\Delta(\phi\tilde{T}_n v)||_{2;\Omega^*(n_0)} \le M(n_0)$. Applying [23, Theorem 8.12] yields $\phi\tilde{T}_n v \in W^{2,2}(\Omega^*(n_0))$ for all $n \ge n_0$ and

$$||\phi\tilde{T}_n v||_{2,2;\Omega^*(n_0)} \le M(\partial D)(||\phi\tilde{T}_n v||_{2;\Omega^*(n_0)} + ||-\Delta(\phi\tilde{T}_n v)||_{2;\Omega^*(n_0)}) \le M(n_0, \partial D).$$

Hence $\phi\tilde{T}_n v$ is a bounded sequence in $W^{2,2}(\Omega^*(n_0))$ and there exists a subsequence, $\phi\tilde{T}_{n_k} v$, such that $\phi\tilde{T}_{n_k} v \xrightarrow{w} u$ in $W^{2,2}(\Omega^*(n_0))$ for some $u \in W^{2,2}(\Omega^*(n_0))$. But $W^{2,2}(\Omega^*(n_0))$ is compactly embedded in $C(\overline{\Omega^*(n_0)})$ thus $\phi\tilde{T}_{n_k} v \to u$ in $C(\overline{\Omega^*(n_0)})$ and, in particular $\tilde{T}_{n_k} v \to u$ pointwise in $\overline{\Omega^*(n_1)}$.

By a diagonalisation process, for $\mathbf{x} \in \Omega$ define $\overline{T}v(\mathbf{x}) = u(\mathbf{x})$. Applying the dominated convergence theorem

$$T_n v(\mathbf{x}) = \int_{\Omega(n_0)} g_n(\mathbf{x}, \mathbf{y}) v(\mathbf{y}) d\mathbf{y} \to \int_{\Omega(n_0)} g(\mathbf{x}, \mathbf{y}) v(\mathbf{y}) d\mathbf{y} = Tv(\mathbf{x})$$

for almost all $\mathbf{x} \in \Omega$. We conclude that

$$\overline{T}v(\mathbf{x}) = \int_{\Omega} g(\mathbf{x}, \mathbf{y}) v(\mathbf{y}) d\mathbf{y}$$

for almost all $\mathbf{x} \in \Omega$.

Let $\tilde{T}v$ denote the extension of Tv, as an odd function of x_2, to a function on Ω^*. Since $\tilde{T}v = 0$ on ∂D and ∂D is of class C^2 we can apply [23, Theorem 9.13 and Lemma 9.16] to obtain $\tilde{T}v \in W^{2,p}(\Omega^*(n_1))$ and for $n > n_1$

$$\begin{aligned} ||\tilde{T}v||_{2,p;\Omega^*(n_1)} &\le M(p, \partial D, \Omega^*(n_1), \Omega^*(n))(||\tilde{T}v||_{p;\Omega^*(n)} + ||\tilde{v}||_{p;\Omega^*(n)}) \\ &\le M(p, \partial D, \Omega^*(n_1), \Omega^*(n), n)||v||_p. \end{aligned} \tag{2.3.3}$$

It follows that $T : L^p(U) \to W^{2,p}(U)$ is bounded and the compactness of the embedding $W^{2,p}(U) \to L^q(U)$ gives the required result. □

We state some inequalities which shall be used frequently. By an application of the maximum principle

$$g(\mathbf{x}, \mathbf{y}) \le g_0(\mathbf{x}, \mathbf{y}) \quad \forall \mathbf{x}, \mathbf{y} \in \Omega, \tag{2.3.4}$$

$$g_1(\mathbf{x}, \mathbf{y}) \le g(\mathbf{x}, \mathbf{y}) \le g_0(\mathbf{x}, \mathbf{y}) \quad \forall \mathbf{x}, \mathbf{y} \in \Theta. \tag{2.3.5}$$

and

$$h_0(\mathbf{x},\mathbf{y}) \leq h(\mathbf{x},\mathbf{y}) \leq h_1(\mathbf{x},\mathbf{y}) \qquad \forall \mathbf{x},\mathbf{y} \in \Theta. \tag{2.3.6}$$

Again, we follow Turkington [36, 37] in defining

$$\begin{aligned} \hat{h}(\mathbf{x},\mathbf{y}) &= h(\mathbf{x},\mathbf{y}) - h_0(\mathbf{x},\mathbf{y}) \qquad \forall \mathbf{x},\mathbf{y} \in \Omega, & (2.3.7)\\ \hat{h}_1(\mathbf{x},\mathbf{y}) &= h_1(\mathbf{x},\mathbf{y}) - h_0(\mathbf{x},\mathbf{y}) \qquad \forall \mathbf{x},\mathbf{y} \in \Theta. & (2.3.8) \end{aligned}$$

We note

$$\hat{h}_1(\mathbf{x},\mathbf{y}) = \frac{1}{4\pi}\log\left(1 + \frac{4x_2y_2}{|\mathbf{y}|^2|\mathbf{x}-|\mathbf{y}|^{-2}\mathbf{y}|^2}\right) \tag{2.3.9}$$

and clearly

$$0 \leq \hat{h}(\mathbf{x},\mathbf{y}) \leq \hat{h}_1(\mathbf{x},\mathbf{y}) \leq \frac{x_2y_2}{\pi(|\mathbf{x}||\mathbf{y}|-1)^2} \quad \forall \mathbf{x},\mathbf{y} \in \Theta. \tag{2.3.10}$$

Also, $\hat{h}(\mathbf{x},\mathbf{y})$ and $\hat{h}_1(\mathbf{x},\mathbf{y})$ are zero when $x_2 = 0$ and can therefore be extended as odd functions of x_2 to harmonic functions $\hat{h}^*(\mathbf{x},\mathbf{y})$ and $\hat{h}_1^*(\mathbf{x},\mathbf{y})$ defined on Ω^* and Θ^* respectively, where $\Theta^* = \{\mathbf{x} \in \mathbb{R}^2 \,|\, |\mathbf{x}| > 1\}$.

LEMMA 2.3.3 *There exists a unique harmonic function η continuous on $\overline{\Omega}$ with $\eta = 0$ on $\partial\Omega$ and*

$$\begin{aligned} \eta(\mathbf{x}) &= x_2 + O(|\mathbf{x}|^{-1}), & (2.3.11)\\ \nabla\eta(\mathbf{x}) &= (0,1) + O(|\mathbf{x}|^{-2}) \qquad \text{as } |\mathbf{x}| \to \infty. & (2.3.12) \end{aligned}$$

Also,

$$x_2 - \frac{x_2}{|\mathbf{x}|^2} \leq \eta(\mathbf{x}) \leq x_2, \qquad \forall \mathbf{x} \in \Theta.$$

Proof Let η_n be the unique harmonic function satisfying $\eta_n = 0$ on ∂D and $\eta_n = x_2$ on $\partial\Omega(n)\backslash\partial D$. By the maximum principle, for $n \geq m$

$$x_2 - \frac{x_2}{|\mathbf{x}|^2} \leq \eta_m(\mathbf{x}) \leq \eta_n(\mathbf{x}) \leq x_2, \qquad \forall \mathbf{x} \in \Theta(m). \tag{2.3.13}$$

Extend η_n, as an odd function of x_2 to a harmonic function η_n^* defined on $\Omega^*(n)$. For $\mathbf{x} \in \Omega^*$ define

$$\eta^*(\mathbf{x}) = \lim_{n\to\infty} \eta_n^*(\mathbf{x}).$$

Then $\Delta\eta^* = 0$ in Ω^*, η^* is continuous onto the boundary of Ω^* and $\eta^* = 0$ on $\partial\Omega^*$.

For $\mathbf{x} \in \Omega^*$ define $u(\mathbf{x}) = x_2 - \eta^*$. By (2.3.13) $|u(\mathbf{x})| \leq |\mathbf{x}|^{-1}$ if $|\mathbf{x}| > 1$. Let $\mathbf{x} \in \Omega$ with $|\mathbf{x}| > 2$ and $B = \{\mathbf{z} \in \Omega^* | |\mathbf{x} - \mathbf{z}| < |\mathbf{x}|/2\}$. Then

$$|\nabla u(\mathbf{x})| \leq \frac{4}{|\mathbf{x}|} \sup_{\mathbf{z}\in\partial B} |u(\mathbf{z})| \leq \frac{8}{|\mathbf{x}|^2}.$$

Hence $\nabla u(\mathbf{x}) = O(|\mathbf{x}|^{-2})$ as $|\mathbf{x}| \to \infty$. Obviously the restriction of η^* to Ω has the required properties. □

LEMMA 2.3.4 *Let $2 < p < \infty$. For $v \in L^p(\Omega)$ with bounded support $Tv = O(|\mathbf{x}|^{-1})$ and $\nabla Tv = O(|\mathbf{x}|^{-2})$ as $|\mathbf{x}| \to \infty$.*

Proof Let U be the support of v and let V consist of all points of Ω with distance at most 1 from U. Then $g(\mathbf{x},\mathbf{y}) = g_0(\mathbf{x},\mathbf{y}) - \hat{h}(\mathbf{x},\mathbf{y})$ where $0 \leq \hat{h}(\mathbf{x},\mathbf{y}) \leq g_0(\mathbf{x},\mathbf{y}) = O(|\mathbf{x}|^{-1})$ as $|\mathbf{x}| \to \infty$ uniformly over $\mathbf{y} \in U$. Hence $Tv(\mathbf{x}) = O(|\mathbf{x}|^{-1})$ as $|\mathbf{x}| \to \infty$.

Let $\tilde{T}v$ denote the extension of Tv, as an odd function of x_2, to a function on Ω^*. Let $M_1, M_2 > 1$ be such that $|\tilde{T}v(\mathbf{x})| \leq M_2/|\mathbf{x}|$ for all $|\mathbf{x}| > M_1$ and $\tilde{T}v$ is harmonic outside $B_{M_1}(0)$. Then for $\mathbf{x} \in \Omega$ with $|\mathbf{x}| > 2M_1$ we have

$$|\nabla\tilde{T}v(\mathbf{x})| \leq \frac{4}{|\mathbf{x}|} \sup_{\mathbf{z}\in\partial B} |\tilde{T}v(\mathbf{z})| \leq \frac{8M_2}{|\mathbf{x}|^2}$$

where $B = B_{|\mathbf{x}|/2}(\mathbf{x})$. □

From Lemmas 2.3.2 and 2.3.4 we obtain the following result which is analogous to Lemma 2.3.1(vi) and proved using the same method.

LEMMA 2.3.5 *Let $2 < p < \infty$ and let $v \in L^p(\Omega)$ have bounded support. Then*

$$\int_\Omega |\nabla Tv|^2 = \int_\Omega vTv < \infty$$

2.4 Vortex pairs when there is no obstacle

In this section we prove the existence of planar flows in which there is no obstacle and give an alternative proof of [11, Theorem 16(i)]. We use estimates for the diameter of the support of the maximisers to determine asymptotic properties of the maximisers. The asymptotic properties are required in Section 2.5 in proving the existence of flows past an obstacle.

2.4.1 Reformulation of the variational problem

Let $2 < p < \infty$ and let q denote the conjugate exponent of p. Let $\omega_0 \in L^p(\Pi)$ be a non-zero non-negative function having bounded support with $||\omega_0||_1 = 1$ and let $\mathcal{G}$ be the set of rearrangements of ω_0 on Π having bounded support. Let $a > 0$ be such that $\mu_2\{\omega_0 > 0\} = \pi a^2$.

Let $\lambda > 0$. For $v \in L^p(\Pi)$ having bounded support, define

$$E_\lambda(v) = \frac{1}{2}\int_\Pi vT_0v - \lambda\int_\Pi x_2 v$$

where $T_0 v$ is as defined in 2.3.1. By Lemma 2.3.1(iv) and Lemma 2.3.1(vi), $T_0 : L^p(\Pi(\xi)) \to L^q(\Pi(\xi))$ is a compact, symmetric strictly positive operator and therefore E_λ is a weakly sequentially continuous strictly convex functional on $L^p(\Pi(\xi))$.

For $\xi > 0$, let $\mathcal{G}(\xi)$ be the set of functions in $\mathcal{G}$ that vanish outside $\Pi(\xi)$ and let $\mathcal{G}^\sharp(\xi)$ denote the set of funcions in $\mathcal{G}(\xi)$ that are Steiner-symmetric.

For $\lambda > 0$ let $c = 1/\lambda$. Then, if $c\xi \geq 2a$, it follows from [9, Theorem 7] that E_λ attains a maximum relative to $\mathcal{G}(c\xi)$ and, in particular, by Lemma 2.3.1(v) there is a maximiser, $\omega^\sharp_{\lambda,\xi/\lambda}$, that is also a member of $\mathcal{G}^\sharp(c\xi)$. By [9, Theorem 7]

$$\omega^\sharp_{\lambda,\xi/\lambda} = \phi_{\lambda,\xi/\lambda} \circ (T_0\omega^\sharp_{\lambda,\xi/\lambda} - \lambda x_2) \tag{2.4.1}$$

almost everywhere in $\Pi(c\xi)$ for some increasing function $\phi_{\lambda,\xi/\lambda}$.

For $v \in L^p(\Pi(c\xi))$ let $v_c(\mathbf{x}) = c^2 v(c\mathbf{x})$. Then $v_c \in L^p(\Pi(\xi))$, $||v_c||_1 = ||v||_1$ and $c^2\mu_2\{v_c > 0\} = \mu_2\{v > 0\}$.

For $v \in L^p(\Pi)$ having bounded support define

$$\begin{aligned} E(v) &= \frac{1}{2}\int_\Pi vT_0v - \int_\Pi x_2 v \\ &= \frac{1}{4\pi}\int_\Pi g_0(\mathbf{x},\mathbf{y})v(\mathbf{x})v(\mathbf{y})d\mathbf{x}d\mathbf{y} - \int_\Pi x_2 v(\mathbf{y})d\mathbf{y}. \end{aligned}$$

For $\mathbf{x} \in \Pi$, let $\zeta_c(\mathbf{x}) = c^2\omega_0(c\mathbf{x})$. Let $\mathcal{G}_c(\xi)$ denote the set of functions that are rearrangements of ζ_c vanishing outside $\Pi(\xi)$ and let $\mathcal{G}_c^\sharp(\xi)$ denote the set of functions in $\mathcal{G}_c(\xi)$ that are Steiner-symmetric.

Then $E(v_c) = E_\lambda(v)$ for all $v \in L^p(\Pi(c\xi))$ and if $c\xi \geq 2a$, E attains a maximum relative to $\mathcal{G}_c(\xi)$. In particular, there is a maximiser, $\zeta_{c,\xi}^\sharp$, that is also a member of $\mathcal{G}_c^\sharp(\xi)$.

Note that $\zeta_{c,\xi}^\sharp$ maximises E relative to $\mathcal{G}_c(\xi)$ if and only if there is a maximiser, $\omega_{\lambda,\xi/\lambda}^\sharp$, of E_λ relative to $\mathcal{G}(c\xi)$ with

$$\omega_{\lambda,\xi/\lambda}^\sharp(\mathbf{x}) = \frac{1}{c^2}\zeta_{c,\xi}^\sharp\left(\frac{\mathbf{x}}{c}\right). \tag{2.4.2}$$

2.4.2 Existence of maximisers of E_λ relative to $\mathcal{G}$

In this section we give an alternative proof of [11, Theorem 16(i)]. We show that for λ sufficiently small E attains a maximum relative to $\mathcal{G}_c$ and therefore E_λ attains a maximum relative to $\mathcal{G}$.

We obtain an estimate for the diameter of the support of $\zeta_{c,\xi}^\sharp$ and from this deduce the existence of ξ_0, C_0 such that for all $c \geq C_0, \xi \geq \xi_0$, $\zeta_{c,\xi_0}^\sharp$ maximises E relative to $\mathcal{G}_c(\xi)$. The method used to obtain an estimate for the diameter of the support of $\zeta_{c,\xi}^\sharp$ does not use the Steiner-symmetry of $\zeta_{c,\xi}^\sharp$ and will be used in Section 2.5 where the techniques of Steiner-symmetry are unavailable. It is similar to the method used by Turkington [36, 37].

For $\mathbf{x} \in \Pi$ define

$$H_0(\mathbf{x}) = \frac{1}{2}h_0(\mathbf{x},\mathbf{x}) + x_2 = \frac{1}{4\pi}\log\frac{1}{2x_2} + x_2. \tag{2.4.3}$$

Let $\hat{\mathbf{X}} = (0, 1/4\pi) = (0, \hat{X}_2)$. Note that H_0 is minimised on Π when $x_2 = 1/4\pi$.

LEMMA 2.4.1 *Let $\xi \geq 1$, $c > 4\pi a$. Then*

$$E(\zeta^\sharp_{c,\xi}) \geq \frac{1}{4\pi}\log\frac{c}{2a} - H_0(\hat{\mathbf{X}}) + o(1) \text{ as } c \to \infty$$

and the rate of convergence is independent of ξ.

Proof For $c > 4\pi a$ let $\hat{\zeta}_c$ be the circular symmetric decreasing rearrangement of ζ_c relative to $\hat{\mathbf{X}}$. Then

$$\begin{aligned}
E(\zeta^\sharp_{c,\xi}) &\geq E(\hat{\zeta}_c) \\
&= \frac{1}{4\pi}\int_\Pi\int_\Pi \log\frac{1}{|\mathbf{x}-\mathbf{y}|}\hat{\zeta}_c(\mathbf{x})\hat{\zeta}_c(\mathbf{y}) - \int_\Pi\int_\Pi \frac{1}{4\pi}\log\frac{1}{|\mathbf{x}-\bar{\mathbf{y}}|}\hat{\zeta}_c(\mathbf{x})\hat{\zeta}_c(\mathbf{y}) \\
&\quad - \int_\Pi x_2\hat{\zeta}_c(\mathbf{x})d\mathbf{x} \\
&\geq \frac{1}{4\pi}\log\left(\frac{c}{2a}\right)||\hat{\zeta}_c||_1^2 - \int_\Pi\int_\Pi \frac{1}{4\pi}\log\frac{1}{|\mathbf{x}-\bar{\mathbf{y}}|}\hat{\zeta}_c(\mathbf{x})\hat{\zeta}_c(\mathbf{y}) \\
&\quad - \int_\Pi x_2\hat{\zeta}_c(\mathbf{x})dx \qquad (2.4.4)
\end{aligned}$$

since supp $\hat{\zeta}_c = B_{a/c}(\hat{\mathbf{X}})$. Also, since $||\hat{\zeta}_c||_1 = 1$

$$\begin{aligned}
&\left|\int_\Pi\int_\Pi \frac{1}{4\pi}\log\frac{1}{|\mathbf{x}-\bar{\mathbf{y}}|}\hat{\zeta}_c(\mathbf{x})\hat{\zeta}_c(\mathbf{y})d\mathbf{x}d\mathbf{y} - \frac{1}{4\pi}\log\frac{1}{2\hat{X}_2}\right| \\
&= \left|\int_\Pi\int_\Pi \left(\frac{1}{4\pi}\log\frac{1}{|\mathbf{x}-\bar{\mathbf{y}}|} - \frac{1}{4\pi}\log\frac{1}{2\hat{X}_2}\right)\hat{\zeta}_c(\mathbf{x})\hat{\zeta}_c(\mathbf{y})d\mathbf{x}d\mathbf{y}\right| \\
&\leq \sup_{\mathbf{x},\mathbf{y}\in B_{a/c}(\hat{\mathbf{X}})} \frac{1}{4\pi}\left|\log\left(\frac{2\hat{X}_2}{|\mathbf{x}-\bar{\mathbf{y}}|}\right)\right| ||\hat{\zeta}_c||_1^2 \\
&\to 0 \quad \text{as } c \to \infty
\end{aligned}$$

Finally

$$\begin{aligned}
\left|\int_\Pi x_2\hat{\zeta}_c(\mathbf{x})d\mathbf{x} - \hat{X}_2\right| &= \left|\int_\Pi (x_2 - \hat{X}_2)\hat{\zeta}_c(\mathbf{x})d\mathbf{x}\right| \\
&\leq \sup_{\mathbf{x}\in B_{a/c}(\hat{\mathbf{X}})} |x_2 - \hat{X}_2| ||\hat{\zeta}_c||_1 \\
&\to 0 \quad \text{as } c \to \infty.
\end{aligned}$$

Letting $c \to \infty$ in (2.4.4) gives the required result. We note that the rate of convergence is independent of ξ. $\square$

For $B \subset \mathbb{R}^2$

$$\operatorname{diam}(B) = \sup\{|\mathbf{x} - \mathbf{y}| | \mathbf{x}, \mathbf{y} \in B\}.$$

DEFINITION 2.4.2 *The "essential diameter" of a bounded, measurable set $U \subset \mathbb{R}^2$ is*

$$\begin{aligned} \textit{ess diam}(U) \quad = \quad & \min\{d | U = U' \cup N \textit{ for some } U', N \subset \Pi(\xi) \textit{ with } \textit{diam}(U') = d \\ & \textit{and } \mu_2(N) = 0\} \end{aligned}$$

Let $S_{c,\xi} = \{\mathbf{x} \in \Pi(\xi) | \zeta^\sharp_{c,\xi}(\mathbf{x}) > 0\}$.

LEMMA 2.4.3 *There exist ξ_0, C_0 such that*

$$\textit{ess diam}(S_{c,\xi}) \leq \frac{R(\xi)a}{c} \leq \frac{\sigma \xi a}{c} \qquad \forall \xi \geq \xi_0, c \geq C_0$$

where $\sigma > 0$ is a constant.

Proof By [9, Theorem 7],

$$\zeta^\sharp_{c,\xi} = \phi_{c,\xi} \circ (T_0 \zeta^\sharp_{c,\xi} - x_2) \tag{2.4.5}$$

almost everywhere in $\Pi(\xi)$ for some increasing function $\phi_{c,\xi}$. Hence

$$S_{c,\xi} = \{\mathbf{x} \in \Pi(\xi) | T_0 \zeta^\sharp_{c,\xi}(\mathbf{x}) - x_2 > \gamma_{c,\xi}\}. \tag{2.4.6}$$

except for a set of zero measure. To see this let $L = \{\mathbf{x} \in \Pi(\xi) | T_0 \zeta^\sharp_{c,\xi}(\mathbf{x}) - x_2 = \gamma_{c,\xi}\}$. Then by [23, Lemma 7.7], $\zeta^\sharp_{c,\xi}(\mathbf{x}) = -\Delta T_0 \zeta^\sharp_{c,\xi}(\mathbf{x}) = 0$ for almost all $\mathbf{x} \in L$.

If $\gamma_{c,\xi} < 0$ then

$$0 < x_2 < |\gamma_{c,\xi}| \Rightarrow T_0 \zeta^\sharp_{c,\xi}(\mathbf{x}) - x_2 > T_0 \zeta^\sharp_{c,\xi}(\mathbf{x}) - |\gamma_{c,\xi}| > \gamma_{c,\xi}.$$

By considering the area of $S_{c,\xi}$ it follows that $\exists C_1, \xi_1$ such that $\gamma_{c,\xi} \geq -1/2$ for all $c \geq C_1, \xi \geq \xi_1$.

Now consider

$$
\begin{aligned}
F(\zeta^{\sharp}_{c,\xi}) &:= \frac{1}{2}\int_{\Pi(\xi)}(T_0\zeta^{\sharp}_{c,\xi} - x_2 - \gamma_{c,\xi})\zeta^{\sharp}_{c,\xi} \\
&\leq \frac{1}{2}\int_{\Pi(\xi)}(T_0\zeta^{\sharp}_{c,\xi} - x_2 - \gamma_{c,\xi} - 1)^{+}\zeta^{\sharp}_{c,\xi} + \frac{1}{2}\int_{\Pi(\xi)}\zeta^{\sharp}_{c,\xi}. \qquad (2.4.7)
\end{aligned}
$$

Let $u = T_0\zeta^{\sharp}_{c,\xi} - x_2 - \gamma_{c,\xi} - 1$. By Lemma 2.3.1(vi) $T_0\zeta^{\sharp}_{c,\xi}(\mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$ hence $u^{+} = \max\{u, 0\} \in H^1_0(\Pi(M))$ for some $M = M_{c,\xi} \in \mathbb{R}$.

From [23, Lemma 7.6] and the Divergence Theorem [24, Theorem 1.5.1] we obtain

$$
\begin{aligned}
\int_{\Pi(\xi)}|\nabla u^{+}|^2 \leq \int_{\Pi(M)}|\nabla u^{+}|^2 &= \int_{\Pi(M)}\nabla u^{+}.\nabla u \\
&= \int_{\Pi(M)} u^{+}\zeta^{\sharp}_{c,\xi} \\
&= \int_{\Pi(\xi)} u^{+}\zeta^{\sharp}_{c,\xi} \\
&\leq ||\zeta^{\sharp}_{c,\xi}||_2\left(\int_{\Pi(\xi)}|u^{+}|^2\right)^{1/2} \\
&= c||\omega_0||_2\left(\int_{\Pi(\xi)}|u^{+}|^2\right)^{1/2}. \qquad (2.4.8)
\end{aligned}
$$

By [1, Lemma 5.14] $W^{1,1}(\Pi(\xi)) \to L^2(\Pi(\xi))$ and the embedding constant depends only on the cone determining the cone property for $\Pi(\xi)$. Hence

$$
\left(\int_{\Pi(\xi)}|u^{+}|^2\right)^{1/2} \leq k\left(\int_{\Pi(\xi)}|u^{+}| + \left|u^{+}_{x_1}\right| + \left|u^{+}_{x_2}\right|\right) \qquad (2.4.9)
$$

for some embedding constant k independent of ξ.

An application of Hölder's inequality yields

$$
\begin{aligned}
\left(\int_{\Pi(\xi)}|u^{+}|^2\right)^{1/2} &\leq k\left(\int_{S_{c,\xi}}d\mu_2\right)^{1/2}\left(\int_{\Pi(\xi)}|u^{+}|^2\right)^{1/2} + k\left(\int_{\Pi(\xi)}\left|u^{+}_{x_1}\right| + \left|u^{+}_{x_2}\right|\right) \\
&= k\left(\frac{\pi a^2}{c^2}\right)^{1/2}\left(\int_{\Pi(\xi)}|u^{+}|^2\right)^{1/2} + k\left(\int_{\Pi(\xi)}\left|u^{+}_{x_1}\right| + \left|u^{+}_{x_2}\right|\right).
\end{aligned}
$$

If $c > 2ka\sqrt{\pi}$ then

$$\begin{aligned}\left(\int_{\Pi(\xi)} |u^+|^2\right)^{1/2} &\le 2k\left(\int_{\Pi(\xi)} \left|u^+_{x_1}\right| + \left|u^+_{x_2}\right|\right)\\ &\le 4k\left(\frac{\pi a^2}{c^2}\right)^{1/2}\left(\int_{\Pi(\xi)} \left|u^+_{x_1}\right|^2 + \left|u^+_{x_2}\right|^2\right)^{1/2} \qquad (2.4.10)\end{aligned}$$

since $\{\mathbf{x} \in \Pi(\xi) | |\nabla u^+(\mathbf{x})| > 0\} \subset S_{c,\xi}$ except for a set of zero measure.

Combining (2.4.8) and (2.4.10)

$$\int_{\Pi(\xi)} |\nabla u^+|^2 \le 4ka\sqrt{\pi}||\omega_0||_2\left(\int_{\Pi(\xi)} |\nabla u^+|^2\right)^{1/2}$$

and therefore

$$\int_{\Pi(\xi)} |\nabla u^+|^2 \le \beta$$

where $\beta = 16k^2\pi a^2||\omega_0||_2^2$ is a constant independent of c and ξ. In particular,

$$\int_{\Pi(\xi)} u^+\zeta^\sharp_{c,\xi} \le 4ka\sqrt{\pi}||\omega_0||_2\left(\int_{\Pi(\xi)} |\nabla u^+|^2\right)^{1/2} \le \beta$$

for all $c \ge C_2 = \max\{C_1, 2ka\sqrt{\pi}\}, \xi \ge \xi_1$. From (2.4.7) we have

$$F(\zeta^\sharp_{c,\xi}) \le \frac{1}{2}(\beta + 1). \qquad (2.4.11)$$

We observe that

$$\begin{aligned}\gamma_{c,\xi} &= 2E(\zeta^\sharp_{c,\xi}) - 2F(\zeta^\sharp_{c,\xi}) + \int_\Pi x_2\zeta^\sharp_{c,\xi} \qquad (2.4.12)\\ &\ge 2E(\zeta^\sharp_{c,\xi}) - 2F(\zeta^\sharp_{c,\xi}).\end{aligned}$$

By Lemma 2.4.1 and (2.4.11) there exists $C_3 \ge \max\{C_2, 4\pi a\}$ such that

$$\gamma_{c,\xi} \ge \frac{1}{2\pi}\log\frac{c}{2a} - \tilde{\beta} \qquad (2.4.13)$$

for all $c \ge C_3, \xi \ge \xi_1$ where $\tilde{\beta} > 0$ is a constant independent of c and ξ.

Let $R \ge 1$. For $\mathbf{x} \in S_{c,\xi}$ let $B = \{\mathbf{y} \in \Pi(\xi) : |\mathbf{y} - \mathbf{x}| < Ra/c\}$. Then by

(2.4.6) and (2.4.13)

$$T_0\zeta^\sharp_{c,\xi}(\mathbf{x}) - \frac{1}{2\pi}\log\frac{c}{2a} > x_2 - \tilde{\beta}$$

for almost all $\mathbf{x} \in S_{c,\xi}$. Hence

$$\int_B \log\left(\frac{2a|\mathbf{x}-\bar{\mathbf{y}}|}{c|\mathbf{x}-\mathbf{y}|}\right)\zeta^\sharp_{c,\xi}(\mathbf{y})d\mathbf{y} + \int_{\Pi(\xi)\backslash B}\log\left(\frac{2a|\mathbf{x}-\bar{\mathbf{y}}|}{c|\mathbf{x}-\mathbf{y}|}\right)\zeta^\sharp_{c,\xi}(\mathbf{y})d\mathbf{y} \geq 2\pi(x_2 - \tilde{\beta}). \tag{2.4.14}$$

for almost all $\mathbf{x} \in S_{c,\xi}$.

It is shown in Appendix A that there exist positive constants $M_1, M_2, M_3 > 0$ independent of c and ξ such that

$$\int_B \log\left(\frac{2a|\mathbf{x}-\bar{\mathbf{y}}|}{c|\mathbf{x}-\mathbf{y}|}\right)\zeta^\sharp_{c,\xi}(\mathbf{y})d\mathbf{y} \leq \begin{cases} (M_1 + M_2|\log x_2|)||\zeta_0||_p & \text{if } x_2 \geq a, \\ M_3||\zeta_0||_p & \text{if } 0 < x_2 \leq a. \end{cases} \tag{2.4.15}$$

Note that for $\mathbf{y} \in \Pi(\xi)\backslash B$ we have $|\mathbf{x}-\mathbf{y}| \geq Ra/c$ and $|\mathbf{x}-\bar{\mathbf{y}}| \leq 2\xi$ hence

$$\int_{\Pi(\xi)\backslash B}\log\left(\frac{2a|\mathbf{x}-\bar{\mathbf{y}}|}{c|\mathbf{x}-\mathbf{y}|}\right)\zeta^\sharp_{c,\xi}(\mathbf{y})d\mathbf{y} \leq \log(\frac{4\xi}{R})\int_{\Pi(\xi)\backslash B}\zeta^\sharp_{c,\xi}(\mathbf{y})d\mathbf{y}.$$

Then rearranging (2.4.14) and applying the estimate (2.4.15)

$$\log(\frac{R}{4\xi})\int_{\Pi(\xi)\backslash B}\zeta^\sharp_{c,\xi}(\mathbf{y})d\mathbf{y} \leq \begin{cases} (M_1 + M_2|\log x_2|)||\zeta_0||_p + 2\pi(\tilde{\beta} - x_2) & \text{if } x_2 \geq a, \\ M_3||\zeta_0||_p + 2\pi(\tilde{\beta} - x_2) & \text{if } 0 < x_2 \leq a \end{cases} \tag{2.4.16}$$

hence

$$\log(\frac{R}{4\xi})\int_{\Pi(\xi)\backslash B}\zeta^\sharp_{c,\xi}(\mathbf{y})d\mathbf{y} < K \tag{2.4.17}$$

where K is a constant independent of c and ξ.

Let $R(\xi) = 4\xi e^{2K}$ and suppose to seek a contradiction that ess diam$(S_{c,\xi}) > 2R(\xi)a/c$. By (2.4.17)

$$\int_{\Pi(\xi)\backslash B}\zeta^\sharp_{c,\xi}(\mathbf{y})d\mathbf{y} < \frac{1}{2} \tag{2.4.18}$$

for almost all $\mathbf{x} \in S_{c,\xi}$. Hence $S_{c,\xi} = S'_{c,\xi} \cup N$ where diam$(S'_{c,\xi}) > 2R(\xi)a/c$, $\mu_2(N) = 0$ and (2.4.18) holds for all $\mathbf{x} \in S'_{c,\xi}$. Choose $\mathbf{x}', \mathbf{x}'' \in S'_{c,\xi}$ such that

$B_{R(\xi)a/c}(\mathbf{x}') \cap B_{R(\xi)a/c}(\mathbf{x}'') = \emptyset$. Then

$$1 \leq \int_{\Pi(\xi)\setminus B_{R(\xi)a/c}(\mathbf{x}')} \zeta^{\sharp}_{c,\xi}(\mathbf{y})d\mathbf{y} + \int_{\Pi(\xi)\setminus B_{R(\xi)a/c}(\mathbf{x}'')} \zeta^{\sharp}_{c,\xi}(\mathbf{y})d\mathbf{y} < \frac{1}{2} + \frac{1}{2} = 1$$

which is a contradiction. □

THEOREM 2.4.4 *There exist* ξ_0, C_0 *such that for all* $\xi \geq \xi_0$, $c \geq C_0$

$$S_{c,\xi} \subset \Pi(\xi_0)$$

except for a set of measure zero.

Proof Let σ, ξ_0, C_0 be as in Lemma 2.4.3. Fix $c > C_1 = \max\{C_0, 8\pi a, 4\sigma a\}$. Let

$$d_\xi = min\{\xi' : S_{c,\xi} \subseteq \Pi(\xi') \text{ except for a set of measure zero}\}$$

and suppose there exists a sequence $\xi_n \to \infty$ such that $d_{\xi_n} \to \infty$. Choose n_0 such that $d_{\xi_n} > \xi_0$, $\forall n \geq n_0$. Then $\zeta^{\sharp}_{c,\xi_n}$ maximises E relative to $\mathcal{F}_c(d_{\xi_n})$ and by the choice of C_1 we have ess diam$(S_{c,\xi_n}) \leq d_{\xi_n}/4$. Hence $S_{c,\xi_n} = S'_{c,\xi_n} \cup N$ where diam$(S'_{c,\xi_n}) \leq d_{\xi_n}/4$, $\mu_2(N) = 0$ and $\zeta^{\sharp}_{c,\xi_n}(\cdot, x_2)$ is a symmetric decreasing function for all $\mathbf{x} \in S'_{c,\xi_n}$. If there exists $\mathbf{x} \in S'_{c,\xi_n}$ with $x_2 \leq d_{\xi_n}/2$ then this contradicts the minimality of d_{ξ_n}. It follows that for all $n \geq n_0$

$$S_{c,\xi_n} \subset \{\mathbf{x} \in \Pi(d_{\xi_n}) | x_2 > d_{\xi_n}/2)\} \tag{2.4.19}$$

except for a set of measure zero.

By (2.4.19) we have for almost all $\mathbf{x}, \mathbf{y} \in S_{c,\xi_n}$,

$$\frac{1}{2}h_0(\mathbf{x},\mathbf{y}) + x_2 = \frac{1}{4\pi}\log\frac{1}{|\mathbf{x}-\bar{\mathbf{y}}|} + x_2 \;\; \geq \;\; \frac{1}{4\pi}\log\frac{1}{2d_{\xi_n}} + \frac{d_{\xi_n}}{2} \tag{2.4.20}$$

$$\to \;\; \infty \text{ as } n \to \infty.$$

Also, $c > 8\pi a$ ensures that supp $\hat{\zeta}_c \subset B_{1/8\pi}(\hat{\mathbf{X}})$ where $\hat{\zeta}_c$ is as in Lemma 2.4.1. Therefore

$$\frac{1}{2}h_0(\mathbf{x},\mathbf{y}) + x_2 = \frac{1}{4\pi}\log\frac{1}{|\mathbf{x}-\bar{\mathbf{y}}|} + x_2 \leq \frac{1}{4\pi}\log 4\pi + \frac{3}{8\pi} \tag{2.4.21}$$

for all $\mathbf{x}, \mathbf{y} \in \operatorname{supp} \hat{\zeta}_c$. But

$$\begin{aligned}
& E(\hat{\zeta}_c) - E(\zeta^\sharp_{c,\xi_n}) \\
= \ & \frac{1}{4\pi}\int_\Pi\int_\Pi \log\frac{1}{|\mathbf{x}-\mathbf{y}|}\hat{\zeta}_c(\mathbf{x})\hat{\zeta}_c(\mathbf{y}) - \frac{1}{4\pi}\int_\Pi\int_\Pi \log\frac{1}{|\mathbf{x}-\mathbf{y}|}\zeta^\sharp_{c,\xi_n}(\mathbf{x})\zeta^\sharp_{c,\xi_n}(\mathbf{y}) \\
& + \int_\Pi\int_\Pi (\frac{1}{2}h_0(\mathbf{x},\mathbf{y}) + x_2)\zeta^\sharp_{c,\xi_n}(\mathbf{x})\zeta^\sharp_{c,\xi_n}(\mathbf{y}) - \int_\Pi\int_\Pi (\frac{1}{2}h_0(\mathbf{x},\mathbf{y}) + x_2)\hat{\zeta}_c(\mathbf{x})\hat{\zeta}_c(\mathbf{y}) \\
\geq \ & \int_\Pi\int_\Pi (\frac{1}{2}h_0(\mathbf{x},\mathbf{y}) + x_2)\zeta^\sharp_{c,\xi_n}(\mathbf{x})\zeta^\sharp_{c,\xi_n}(\mathbf{y}) - \int_\Pi\int_\Pi (\frac{1}{2}h_0(\mathbf{x},\mathbf{y}) + x_2)\hat{\zeta}_c(\mathbf{x})\hat{\zeta}_c(\mathbf{y}) \\
> \ & 0
\end{aligned}$$

if d_{ξ_n} is sufficiently large independently of c by (2.4.20) and (2.4.21) which is a contradiction. □

We immediately obtain [11, Theorem 16(i)] from Theorem 2.4.4.

THEOREM 2.4.5 *There exists $\lambda_0 > 0$ such that if $0 < \lambda < \lambda_0$ then E_λ attains a maximum relative to $\mathcal{F}$. If ω_λ is a maximiser and $\psi_\lambda = T_0\omega_\lambda$, then*

$$-\Delta\psi_\lambda = \phi \circ (\psi_\lambda - \lambda x_2)$$

almost everywhere in Π for some increasing function ϕ.

Proof The proof is exactly as in [11] but is included here for completeness. Recall

$$\omega^\sharp_{\lambda,\xi/\lambda}(\mathbf{x}) = \frac{1}{c^2}\zeta^\sharp_{c,\xi}\left(\frac{\mathbf{x}}{c}\right)$$

and $\omega^\sharp_{\lambda,\xi/\lambda}$ maximises E_λ relative to $\mathcal{G}(\xi/\lambda)$. By Theorem 2.4.4, for $\xi \geq \xi_0$, $0 < \lambda < \lambda_0 = 1/C_0$ the support of $\omega^\sharp_{\lambda,\xi/\lambda}$ is contained in $\Pi(\xi_0/\lambda)$ except for a set of measure zero, hence $\omega^\sharp_{\lambda,\xi_0/\lambda}$ maximises E_λ relative to $\mathcal{G}$.

Write $\psi_\lambda = T\omega^\sharp_{\lambda,\xi_0/\lambda}$. Then $\omega^\sharp_{\lambda,\xi_0/\lambda} = \phi_\lambda \circ (\psi_\lambda - \lambda x_2)$ almost everywhere in $\Pi(\xi_0/\lambda)$ for some increasing function ϕ_λ for which we can assume that $\phi_\lambda(s) \geq 0$ for all $s \in \operatorname{dom} \phi_\lambda$. Since $\omega^\sharp_{\lambda,\xi_0/\lambda}$ is an increasing function of $(\psi_\lambda - \lambda x_2)$ almost everywhere in $\Pi(\xi/\lambda)$ for all $\xi \geq \xi_0$, there exists γ_λ such that $\psi_\lambda - \lambda x_2 \geq \gamma_\lambda$ almost everywhere on $\{\omega^\sharp_{\lambda,\xi/\lambda} > 0\}$ and $\psi_\lambda - \lambda x_2 < \gamma_\lambda$ almost everywhere in

$\Pi \backslash \Pi(\xi_0/\lambda)$. Define

$$\phi(s) = \begin{cases} \phi_\lambda(s) & \text{if } s \in \text{dom } \phi_\lambda,\ s \geq \gamma_\lambda \\ 0 & \text{if } s < \gamma_\lambda \end{cases}$$

Then ϕ is increasing and $\omega^\sharp_{\lambda,\xi_0/\lambda} = \phi \circ (\psi_\lambda - \lambda x_2)$ almost everywhere in Π. $\square$

We show that if λ is sufficiently large then E_λ does not attain a maximum relative to $\mathcal{G}$.

LEMMA 2.4.6 *There exists λ_1 such that for $\lambda > \lambda_1$ we have $\sup_{v \in \mathcal{G}} E_\lambda(v) = 0$ and this supremum is not attained.*

Proof By Lemma 2.3.1(iii) there exists a constant N depending only on a and p such that for all $v \in \mathcal{G}$

$$\begin{aligned} E_\lambda(v) &= \frac{1}{2}\int_\Pi vT_0v - \lambda \int_\Pi x_2 v \\ &\leq \int_\Pi (N||\omega_0||_p/2 - \lambda)x_2 v \\ &< 0 \end{aligned}$$

if $\lambda > N||\omega_0||_p/2$.

Let $\lambda > N||\omega_0||_p/2$. Let v_n denote a rearrangement of ω_0 with bounded support in $\{\mathbf{x} \in \Pi | x_2 < 1/n\}$. Then

$$E_\lambda(v_n) = \frac{1}{2}\int_\Pi v_n T_0 v_n - \lambda \int_\Pi x_2 v_n > -\frac{\lambda||v_n||_1}{n} = -\frac{\lambda||\omega_0||_1}{n}.$$

Hence $\sup_{v \in \mathcal{G}} E_\lambda(v) = 0$ but this supremum is not attained. $\square$

2.4.3 Asymptotic properties of the maximisers

In this section $\zeta^\sharp_c$ will denote a maximiser $\zeta^\sharp_{c,\xi_0}$ of E relative to $\mathcal{G}_c$ and $\omega^\sharp_\lambda$ the corresponding maximiser of E_λ relative to $\mathcal{G}$ as in Theorem 2.4.5. Let $R = R(\xi_0)$ be as in Lemma 2.4.3 and let $\mathbf{X}_c = (0, X_{c2})$ denote the centre of vorticity where $X_{c2} = \int_\Pi x_2 \zeta^\sharp_c$.

In order to be able to work with "sequences" of maximisers we introduce some further terminology.

Let $\mathcal{N}$ be a strictly increasing sequence of real numbers tending to infinity. We shall refer to a sequence $\{u_c\}_{c\in\mathcal{N}}$ by "a sequence u_c" and we write $u_c \to u$ as $c \to \infty$ to mean $u_c \to u$ as $c \to \infty$ through $\mathcal{N}$.

We shall similarly abuse notation by referring to the convergence of "sequences" u_λ as $\lambda \to 0$.

We prove some results which are analogous to those of Turkington [36, Section 4] and the methods used here are very similar to those of Turkington.

LEMMA 2.4.7 *For any sequence* $\mathbf{X}_c$*,*

$$cX_{c2} \to \infty \qquad \textit{as } c \to \infty. \tag{2.4.22}$$

Proof By Lemma (2.4.1)

$$\sup\{E(v)|v \in \mathcal{G}_c(\xi_0)\} \to \infty \tag{2.4.23}$$

as $c \to \infty$ where the rate of convergence is independent of ξ_0. Then for $c \geq C_0$

$$\begin{aligned}\sup\{E(v)|v \in \mathcal{G}_c(\xi_0)\} &\leq ||\zeta_c^\sharp||_1 \sup\{\frac{1}{2}T_0\zeta_c^\sharp(\mathbf{x}) - x_2|\mathbf{x} \in \Pi(\xi_0)\} \\ &\leq \sup\{T_0\zeta_c^\sharp(\mathbf{x}) - x_2|\mathbf{x} \in \Pi(\xi_0)\}.\end{aligned} \tag{2.4.24}$$

Let $r(c,\xi_0) = (0, r_2(c,\xi_0)) \in \overline{\Pi(\xi_0)}$ be such that $T_0\zeta_c^\sharp - x_2$ attains its supremum relative to $\overline{\Pi(\xi_0)}$ at $r(c,\xi_0)$. By Lemma 2.3.1(iii)

$$|T_0\zeta_c^\sharp(\mathbf{x})| = |T_0\omega_\lambda^\sharp(c\mathbf{x})| \leq Ncx_2||\omega_0||_p \tag{2.4.25}$$

where $N > 0$ depends on a and p only. Combining (2.4.23), (2.4.24) and (2.4.25), for each $M > 0$ there exists C_M such that

$$Ncr_2(c,\xi_0)||\omega_0||_p \geq \sup\{T_0\zeta_c^\sharp - x_2|\mathbf{x} \in \Pi(\xi_0)\} \geq MN||\omega_0||_p$$

for all $c \geq C_M$, i.e.

$$r_2(c,\xi_0) \geq \frac{M}{c}$$

for all $c \geq C_M$.

Since $\zeta_c^\sharp$ is essentially an increasing function of $(T_0\zeta_c^\sharp - x_2)$ we have $S_{c,\xi_0} = S'_{c,\xi_0} \cup N$ where $\mathrm{diam}(S'_{c,\xi_0}) \leq \sigma\xi_0 a/c$, $\mu_2(N) = 0$ and $r(c,\xi_0)$ is in the closure of S'_{c,ξ_0}. Let $\mathbf{x}_0 \in S'_{c,\xi_0}$. Then

$$|\mathbf{x}_0 - \mathbf{X}_c| = |\int_\Pi (\mathbf{x}_0 - \mathbf{x})\zeta_c^\sharp(\mathbf{x})d\mathbf{x}| \leq \frac{\sigma\xi_0 a}{c}.$$

and the result follows. □

LEMMA 2.4.8 *Choose a subsequence, $\zeta_c^\sharp$, such that there exists a limiting centre $\mathbf{X} = (0, X_2)$ as $c \to \infty$ (i.e. $|X_2 - X_{c2}| \to 0$). Then $\mathbf{X} = \hat{\mathbf{X}}$.*

Proof

$$\begin{aligned}
& \int_\Pi\int_\Pi \frac{1}{2}h_0(\mathbf{x},\mathbf{y})\zeta_c^\sharp(\mathbf{x})\zeta_c^\sharp(\mathbf{y})d\mathbf{x}d\mathbf{y} + \int_\Pi x_2\zeta_c^\sharp(\mathbf{x})d\mathbf{x} && (2.4.26)\\
= & \int_\Pi\int_\Pi \frac{1}{4\pi}\log\frac{1}{|\mathbf{x}-\mathbf{y}|}\zeta_c^\sharp(\mathbf{x})\zeta_c^\sharp(\mathbf{y})d\mathbf{x}d\mathbf{y} - E(\zeta_c^\sharp)\\
\leq & \int_\Pi\int_\Pi \frac{1}{4\pi}\log\frac{1}{|\mathbf{x}-\mathbf{y}|}\hat{\zeta}_c(\mathbf{x})\hat{\zeta}_c(\mathbf{y})d\mathbf{x}d\mathbf{y} - E(\hat{\zeta}_c)\\
= & \int_\Pi\int_\Pi \frac{1}{2}h_0(\mathbf{x},\mathbf{y})\hat{\zeta}_c(\mathbf{x})\hat{\zeta}_c(\mathbf{y})d\mathbf{x}d\mathbf{y} + \int_\Pi x_2\hat{\zeta}_c(\mathbf{x})d\mathbf{x} && (2.4.27)
\end{aligned}$$

By Lemma 2.4.3 and the methods of Lemma 2.4.1

$$\begin{aligned}
& \left|\int_\Pi\int_\Pi \frac{1}{4\pi}\log\frac{1}{|\mathbf{x}-\bar{\mathbf{y}}|}\zeta_c^\sharp(\mathbf{x})\zeta_c^\sharp(\mathbf{y})d\mathbf{x}d\mathbf{y} - \frac{1}{4\pi}\log\frac{1}{2X_2}\right|\\
= & \left|\int_\Pi\int_\Pi \left(\frac{1}{4\pi}\log\frac{2X_2}{|\mathbf{x}-\bar{\mathbf{y}}|}\right)\zeta_c^\sharp(\mathbf{x})\zeta_c^\sharp(\mathbf{y})d\mathbf{x}d\mathbf{y}\right|\\
\leq & \frac{1}{4\pi}\sup_{\mathbf{x},\mathbf{y}\in B_{Ra/c}(\mathbf{X}_c)}\left|\log\frac{2X_2}{|\mathbf{x}-\bar{\mathbf{y}}|}\right| ||\zeta_c^\sharp||_1^2\\
\to & \ 0 \quad \text{as } c \to \infty.
\end{aligned}$$

Also

$$\begin{aligned}
\left|\int_\Pi x_2\zeta_c^\sharp(\mathbf{x})d\mathbf{x} - X_2\right| & = \left|\int_\Pi (x_2 - X_2)\zeta_c^\sharp(\mathbf{x})d\mathbf{x}\right|\\
& \leq \sup_{\mathbf{x}\in B_{Ra/c}(\mathbf{X}_c)} |x_2 - X_2|\\
& \to 0 \quad \text{as } c \to \infty
\end{aligned}$$

Letting $c \to \infty$ in (2.4.26) and (2.4.27) and using the asymptotic value of (2.4.27) calculated in Lemma 2.4.1 gives

$$H_0(\mathbf{X}) = \frac{1}{4\pi}\log\frac{1}{2X_2} + X_2 \leq \frac{1}{4\pi}\log\frac{1}{2\hat{X}_2} + \hat{X}_2 = H_0(\hat{\mathbf{X}}).$$

But H_0 is minimised on $\Pi(\xi_0)$ when $x_2 = 1/4\pi$. Therefore $\mathbf{X} = \hat{\mathbf{X}}$. □

We state some further asymptotic results. The proofs are similar (but simpler) to those of the analogous results for flows past an obstacle which are included in Section 2.5.

THEOREM 2.4.9 *For any sequence $\zeta_c^\sharp$ and $\mathbf{x} \in B_{Ra}(0)$, define*

$$f_\lambda^\sharp(\mathbf{x}) = \frac{1}{c^2}\zeta_c^\sharp\left(\mathbf{X}_c + \frac{\mathbf{x}}{c}\right).$$

Then $f_\lambda^\sharp \to \hat{\omega}$ in $L^p(B_{Ra}(0))$ as $\lambda \to 0$ where $\hat{\omega}$ is the circularly symmetric decreasing rearrangement of ω_0 relative to the origin.

Define

$$v_\lambda(\mathbf{x}) = (T_0\omega_\lambda^\sharp - \lambda x_2 - \gamma_\lambda)(c\mathbf{X}_c + \mathbf{x}), \quad \text{if } c\mathbf{X}_c + \mathbf{x} \in \Pi,$$

and for $\mathbf{x} \in \mathbb{R}^2$ let

$$V(\mathbf{x}) = \frac{1}{2\pi}\int \log\frac{1}{|\mathbf{x}-\mathbf{y}|}\hat{\omega}(\mathbf{y})d\mathbf{y}.$$

THEOREM 2.4.10 *Let $b = V((0,a))$. Then for $R_1 > R$*

$$v_\lambda \to V - b \quad \textit{in } C^1(\overline{B_{R_1a}(0)})$$

as $\lambda \to 0$.

2.5 Vortex pairs in flows past an obstacle

2.5.1 Reformulation of the variational problem

Let $2 < p < \infty$ and let q denote the conjugate exponent of p. Let $\omega_0 \in L^p(\Omega)$ be a non-zero non-negative function having bounded support with $||\omega_0||_1 = 1$ and let $\mathcal{F}$ be the set of rearrangements of ω_0 on Ω having bounded support. Let $a > 0$ be such that $\mu_2\{\omega_0 > 0\} = \pi a^2$.

Let $\lambda > 0$. For $v \in L^p(\Omega)$ having bounded support, define

$$\Psi_\lambda(v) = \frac{1}{2}\int_\Omega vTv - \lambda \int_\Omega \eta v$$

where η is the unique harmonic function with $\eta = 0$ on $\partial\Omega$ and

$$\begin{aligned} \eta(\mathbf{x}) &= x_2 + O(|\mathbf{x}|^{-1}), & (2.5.28)\\ \nabla\eta(\mathbf{x}) &= (0,1) + O(|\mathbf{x}|^{-2}) \qquad \text{as } |\mathbf{x}| \to \infty. & (2.5.29)\end{aligned}$$

From Lemma 2.3.3 we recall

$$x_2 - \frac{x_2}{|\mathbf{x}|^2} \leq \eta(\mathbf{x}) \leq x_2 \qquad \forall \mathbf{x} \in \Theta. \tag{2.5.30}$$

By Lemmas 2.3.2 and 2.3.5 for a bounded subset U of Ω, the operator $T : L^p(U) \to L^q(U)$ is a compact, symmetric strictly positive operator and therefore Ψ_λ is a weakly sequentially continuous strictly convex functional on $L^p(U)$.

For $\xi > 0$, let $\mathcal{F}(\xi)$ be the set of functions in $\mathcal{F}$ that vanish outside $\Omega(\xi)$. Let $c = 1/\lambda$. Then, if $c\xi \geq 2(a+1)$, there exists a maximiser $\tilde{\omega}_{\lambda,\xi/\lambda}$ of Ψ_λ relative to $\mathcal{F}(c\xi)$. By [9, Theorem 7]

$$\tilde{\omega}_{\lambda,\xi/\lambda} = \phi_{\lambda,\xi/\lambda} \circ (T\tilde{\omega}_{\lambda,\xi/\lambda} - \lambda\eta)$$

almost everywhere in $\Omega(c\xi)$ for some increasing function $\phi_{\lambda,\xi/\lambda}$.

Define $\Omega_c = \{\mathbf{x} \in \Pi | c\mathbf{x} \in \Omega\}$ and let $\Omega_c(\xi) = \{\mathbf{x} \in \Omega_c | |\mathbf{x}| < \xi\}$. For $v \in L^p(\Omega(c\xi))$, define $v_c(\mathbf{x}) = c^2 v(c\mathbf{x})$. Then $v_c \in L^p(\Omega_c(\xi))$, $||v_c||_1 = ||v||_1$ and $||v_c||_p = c^{2/q}||v||_p$. Also $c^2\mu_2\{v_c > 0\} = \mu_2\{v > 0\}$.

For $v \in L^p(\Omega_c)$ having bounded support and $\mathbf{x} \in \Omega_c$ let

$$\begin{aligned} T_c v(\mathbf{x}) &= \int_{\Omega_c} g(c\mathbf{x}, c\mathbf{y}) v(\mathbf{y}) d\mathbf{y},\\ \eta_c(\mathbf{x}) &= \frac{1}{c}\eta(c\mathbf{x}) \end{aligned}$$

and define

$$\tilde{\Psi}_\lambda(v) = \frac{1}{2}\int_{\Omega_c} vT_c v - \int_{\Omega_c} \eta_c v.$$

Then for any bounded subset U of Ω_c, the operator $T_c : L^p(U) \to L^q(U)$ is a compact, symmetric strictly positive operator and therefore $\tilde{\Psi}_\lambda$ is a weakly sequentially continuous strictly convex functional on $L^p(U)$.

Let $\zeta_c(\mathbf{x}) = c^2\omega_0(c\mathbf{x})$. Let $\mathcal{F}_c$ denote the set of rearrangements of ζ_c on Ω_c having bounded support and let $\mathcal{F}_c(\xi)$ denote the set of rearrangements of ζ_c vanishing outside $\Omega_c(\xi)$. Then $\tilde{\Psi}_\lambda(v_c) = \Psi_\lambda(v)$ for all $v \in L^p(\Omega(c\xi))$ and, if $c\xi \geq 2(a+1)$, there exists a maximiser $\tilde{\zeta}_{c,\xi}$ of $\tilde{\Psi}_\lambda$ relative to $\mathcal{F}_c(\xi)$. Note that $\tilde{\zeta}_{c,\xi}$ maximises $\tilde{\Psi}_\lambda$ relative to $\mathcal{F}_c(\xi)$ if and only if there is a maximiser, $\tilde{\omega}_{\lambda,\xi/\lambda}$, of Ψ_λ relative to $\mathcal{F}(\xi/\lambda)$ with

$$\tilde{\omega}_{\lambda,\xi/\lambda}(\mathbf{x}) = \frac{1}{c^2}\tilde{\zeta}_{c,\xi}(\frac{\mathbf{x}}{c}). \tag{2.5.31}$$

2.5.2 Existence of maximisers of Ψ_λ relative to $\mathcal{F}$

We show that for λ sufficiently small $\tilde{\Psi}_\lambda$ attains a maximum relative to $\mathcal{F}_c$ and therefore Ψ_λ attains a maximum relative to $\mathcal{F}$. We obtain an estimate for the diameter of the support of maximisers of $\tilde{\Psi}_\lambda$ relative to functions in $\mathcal{F}_c$ supported on a suitable class of domains. We then deduce that for c and ξ sufficiently large the support of $\tilde{\zeta}_{c,\xi}$ is contained in a ball of fixed radius centred on the x_1-axis. The results from Section 2.4 are used to show that the support of $\tilde{\zeta}_{c,\xi}$ is contained in a fixed ball centred at the origin for c and ξ sufficiently large.

LEMMA 2.5.1 *Let $U \subset \Omega$ be a domain with $B_{1/4\pi}((s, 1/4\pi)) \subset U$ for some $s \in \mathbb{R}$. Let $c \geq 4\pi(a+1)$ and let $\tilde{\zeta}_{c,U}$ denote a maximiser of $\tilde{\Psi}_\lambda$ relative to functions in $\mathcal{F}_c$ that vanish outside U. Then*

$$\tilde{\Psi}_\lambda(\tilde{\zeta}_{c,U}) \geq \frac{1}{4\pi}\log\frac{c}{2a} - H_0(\hat{\mathbf{X}}) + o(1) \text{ as } c \to \infty$$

where $\hat{\mathbf{X}} = (0, 1/4\pi)$ and the rate of convergence is independent of s.

Proof Let $\mathbf{Y}_s = (s, 1/4\pi) = (s, \hat{X}_2)$ and let $\hat{\zeta}_{c,s}$ denote the circularly symmetric decreasing rearrangement of ζ_c relative to $\mathbf{Y}_s$. Then $\hat{\zeta}_{c,s} \in \mathcal{F}_c$ vanishes outside U and

$$\tilde{\Psi}_\lambda(\tilde{\zeta}_{c,U}) \geq \tilde{\Psi}_\lambda(\hat{\zeta}_{c,s})$$

$$
\begin{aligned}
&= \int_\Omega\int_\Omega \left(\frac{1}{4\pi}\log\frac{1}{c|\mathbf{x}-\mathbf{y}|} - \frac{1}{2}h(c\mathbf{x},c\mathbf{y})\right)\hat{\zeta}_{c,s}(\mathbf{x})\hat{\zeta}_{c,s}(\mathbf{y}) - \int_\Omega \eta_c\hat{\zeta}_{c,s} \\
&\geq \frac{1}{4\pi}\log\frac{c}{2a} - \int_\Omega\int_\Omega \left(\frac{1}{2}h(c\mathbf{x},c\mathbf{y}) - \frac{1}{4\pi}\log\frac{1}{c}\right)\hat{\zeta}_{c,s}(\mathbf{x})\hat{\zeta}_{c,s}(\mathbf{y}) \\
&\quad - \int_\Omega \eta_c\hat{\zeta}_{c,s}. \qquad (2.5.32)
\end{aligned}
$$

Since $||\hat{\zeta}_{c,s}||_1 = 1$ we have

$$
\begin{aligned}
&\left|\int_\Omega\int_\Omega \left(\frac{1}{2}h(c\mathbf{x},c\mathbf{y}) - \frac{1}{4\pi}\log\frac{1}{c}\right)\hat{\zeta}_{c,s}(\mathbf{x})\hat{\zeta}_{c,s}(\mathbf{y}) - \frac{1}{4\pi}\log\frac{1}{2\hat{X}_2}\right| \\
&\leq \sup_{\mathbf{x},\mathbf{y}\in B_{a/c}(\mathbf{Y}_s)} \left|\frac{1}{2}h(c\mathbf{x},c\mathbf{y}) - \frac{1}{4\pi}\log\frac{1}{2c\hat{X}_2}\right|.
\end{aligned}
$$

But

$$
\begin{aligned}
\sup_{\mathbf{x},\mathbf{y}\in B_{a/c}(\mathbf{Y}_s)} \left|\frac{1}{2}h_0(c\mathbf{x},c\mathbf{y}) - \frac{1}{4\pi}\log\frac{1}{2c\hat{X}_2}\right| &= \sup_{\mathbf{x},\mathbf{y}\in B_{a/c}(\mathbf{Y}_s)} \left|\frac{1}{4\pi}\log\frac{2\hat{X}_2}{|\mathbf{x}-\bar{\mathbf{y}}|}\right| \\
&\to 0 \quad \text{as } c\to\infty
\end{aligned}
$$

and by (2.3.10)

$$
\begin{aligned}
\sup_{\mathbf{x},\mathbf{y}\in B_{a/c}(\mathbf{Y}_s)} \left|\frac{1}{2}\hat{h}_1(c\mathbf{x},c\mathbf{y})\right| &\leq \sup_{\mathbf{x},\mathbf{y}\in B_{a/c}(\mathbf{Y}_s)} \frac{1}{2\pi}\left|\frac{c^2x_2y_2}{(|\mathbf{y}||\mathbf{x}|c^2-1)^2}\right| \\
&\leq \frac{1}{2\pi}\left|\frac{c^2(\hat{X}_2+\frac{a}{c})^2}{((\hat{X}_2-\frac{a}{c})^2c^2-1)^2}\right| \\
&\to 0 \quad \text{as } c\to\infty.
\end{aligned}
$$

From these estimates and (2.3.6), (2.3.8) we obtain

$$
\sup_{\mathbf{x},\mathbf{y}\in B_{a/c}(\mathbf{Y}_s)} \left|\frac{1}{2}h(c\mathbf{x},c\mathbf{y}) - \frac{1}{4\pi}\log\frac{1}{2c\hat{X}_2}\right| \to 0 \quad \text{as } c\to\infty.
$$

Also, using (2.5.30)

$$
\begin{aligned}
\left|\int_\Omega \eta_c(\mathbf{x})\hat{\zeta}_{c,s}(\mathbf{x})d\mathbf{x} - \hat{X}_2\right| &\leq \sup_{\mathbf{x}\in B_{a/c}(\mathbf{Y}_s)} \left|\eta_c(\mathbf{x}) - \hat{X}_2\right| \\
&\leq \frac{a}{c} + \sup_{\mathbf{x}\in B_{a/c}(\mathbf{Y}_s)} \frac{x_2}{c^2|\mathbf{x}|^2}
\end{aligned}
$$

$$\rightarrow 0 \quad \text{as } c \rightarrow \infty$$

Letting $c \rightarrow \infty$ in (2.5.32) gives the required inequality. Note that the rate of convergence is independent of s. $\square$

LEMMA 2.5.2 *For $\xi \geq 1$ and $c \geq 4\pi(a+1)$ let $U(\xi) = \tilde{U}(\xi) \cap \Omega_c$ where $\tilde{U}(\xi) \subset \Pi$ is convex with*

$$\xi = \mu_1(\partial\tilde{U}(\xi) \cap \partial\Pi) \geq \frac{1}{2} diam(\tilde{U}(\xi)) \tag{2.5.33}$$

and

$$B_{1/4\pi}((s, 1/4\pi)) \subset U(\xi) \text{ for some } s \in \partial\tilde{U}(\xi) \cap \partial\Pi. \tag{2.5.34}$$

Let $\tilde{\zeta}_{c,U(\xi)}$ be a maximiser of $\tilde{\Psi}_\lambda$ relative to functions in $\mathcal{F}_c$ that vanish outside $U(\xi)$ and let $A_{c,U(\xi)} = \{\tilde{\zeta}_{c,U(\xi)} > 0\}$. Then there exists ξ_0 such that

$$ess\ diam(A_{c,U(\xi)}) \leq \frac{R(k,\xi)a}{c} \leq \frac{\sigma(k)\xi a}{c} \qquad \text{whenever } \xi \geq \xi_0, c \geq C(k),$$

where k is a constant depending on the cone which determines the cone property for $\tilde{U}(\xi)$.

Proof We omit the subscripts throughout this proof by writing $\tilde{\zeta} = \tilde{\zeta}_{c,U(\xi)}$ and $A = A_{c,U(\xi)}$ but we emphasize the dependence on c and ξ.

By [9, Theorem 7]

$$\tilde{\zeta} = \phi \circ (T_c\tilde{\zeta} - \eta_c)$$

almost everywhere in $U(\xi)$ for some increasing function $\phi = \phi_{c,U(\xi)}$. Hence

$$A = \{\mathbf{x} \in U(\xi) | T_c\tilde{\zeta}(\mathbf{x}) - \eta_c(\mathbf{x}) > \gamma\} \tag{2.5.35}$$

except for a set of zero measure, for some $\gamma = \gamma_{c,U(\xi)}$. To see this let

$$L = \{\mathbf{x} \in U(\xi) | T_c\tilde{\zeta}(\mathbf{x}) - \eta_c(\mathbf{x}) = \gamma\}.$$

Then by [23, Lemma 7.7], $\tilde{\zeta}(\mathbf{x}) = -\Delta T_c\tilde{\zeta}(\mathbf{x}) = 0$ for almost all $\mathbf{x} \in L$.

If $\gamma < 0$ then

$$0 < x_2 < |\gamma| \Rightarrow T_c\tilde{\zeta}(\mathbf{x}) - \eta_c(\mathbf{x}) > T_c\tilde{\zeta}(\mathbf{x}) - |\gamma| > \gamma.$$

Therefore

$$\frac{\pi a^2}{c^2} = \mu_2\{T_c\tilde{\zeta} - \eta_c > \gamma\} \geq \mu_2\{\mathbf{x} \in U(\xi) | 0 < x_2 < |\gamma|\}$$

and it follows that there exist C_1, ξ_1 such that $\gamma \geq -1/2$ for all $c \geq C_1, \xi \geq \xi_1$.

Now consider

$$\begin{aligned} F(\tilde{\zeta}) &:= \frac{1}{2}\int_{U(\xi)} (T_c\tilde{\zeta} - \eta_c - \gamma)\tilde{\zeta} \\ &\leq \frac{1}{2}\int_{U(\xi)} (T_c\tilde{\zeta} - \eta_c - \gamma - 1)^+\tilde{\zeta} + \frac{1}{2}\int_{U(\xi)} \tilde{\zeta}. \end{aligned} \tag{2.5.36}$$

Let $u = T_c\tilde{\zeta} - \eta_c - \gamma - 1$. By (2.3.4)

$$T_c\tilde{\zeta}(\mathbf{x}) = \int_{\Omega_c} g(c\mathbf{x}, c\mathbf{y})\tilde{\zeta}(\mathbf{y})d\mathbf{y} \leq \int_{\Omega_c} g_0(c\mathbf{x}, c\mathbf{y})\tilde{\zeta}(\mathbf{y})d\mathbf{y} = T_0\tilde{\zeta}(\mathbf{x}) \tag{2.5.37}$$

hence, using the decay estimates from Lemma 2.3.1(vi), $u^+ = max\{u, 0\} \in H_0^1(\Omega_c(M))$ for some $M \in \mathbb{R}$ with $U(\xi) \subset \Omega_c(M)$. Then by [23, Lemma 7.6] and the Divergence Theorem [24, Theorem 1.5.1]

$$\begin{aligned} \int_{U(\xi)} |\nabla u^+|^2 \leq \int_{\Omega_c(M)} |\nabla u^+|^2 &= \int_{\Omega_c(M)} \nabla u^+ . \nabla u \\ &= \int_{U(\xi)} u^+\tilde{\zeta} \\ &\leq ||\tilde{\zeta}||_2 \left(\int_{U(\xi)} |u^+|^2\right)^{1/2} \\ &= c||\omega_0||_2 \left(\int_{U(\xi)} |u^+|^2\right)^{1/2}. \end{aligned} \tag{2.5.38}$$

By [1, Lemma 5.14] for an arbitrary domain $G \subset \mathbb{R}^2$ having the cone property, $W^{1,1}(G) \to L^2(G)$ and the embedding constant depends only on the cone determining the cone property for G. If necessary we may extend u^+ and $\tilde{\zeta}$ to be zero on $\tilde{U}(\xi)\backslash U(\xi)$ and apply the embedding stated above on $\tilde{U}(\xi)$ to obtain the inequality

$$\left(\int_{U(\xi)} |u^+|^2\right)^{1/2} \leq k \left(\int_{U(\xi)} |u^+| + |u^+_{x_1}| + |u^+_{x_2}|\right)$$

for some constant k (independent of c) depending on the cone which determines the cone property for $\tilde{U}(\xi)$.

Since

$$\{\mathbf{x} \in U(\xi) | |\nabla u^+(\mathbf{x})| > 0\} \subset \{\mathbf{x} \in U(\xi) | u(\mathbf{x}) > 0\} \subset A$$

except for sets of zero measure, applying Hölder's inequality yields

$$\left(\int_{U(\xi)} |u^+|^2\right)^{1/2} \leq \frac{k\sqrt{\pi}a}{c}\left(\int_{U(\xi)} |u^+|^2\right)^{1/2} + \frac{2k\sqrt{\pi}a}{c}\left(\int_{U(\xi)} |\nabla u^+|^2\right)^{1/2}$$

If $c > 2k\sqrt{\pi}a$ then

$$\left(\int_{U(\xi)} |u^+|^2\right)^{1/2} \leq \frac{4k\sqrt{\pi}a}{c}\left(\int_{U(\xi)} |\nabla u^+|^2\right)^{1/2}. \tag{2.5.39}$$

Combining (2.5.38) and (2.5.39)

$$\int_{U(\xi)} |\nabla u^+|^2 \leq 4k\sqrt{\pi}a||\omega_0||_2 \left(\int_{U(\xi)} |\nabla u^+|^2\right)^{1/2}$$

and therefore

$$\int_{U(\xi)} |\nabla u^+|^2 \leq \beta(k)$$

where $\beta(k) = 16k^2\pi a^2||\omega_0||_2^2$ is a constant independent of c. In particular,

$$\int_{U(\xi)} u^+\tilde{\zeta} \leq 4ka\sqrt{\pi}||\omega_0||_2 \left(\int_{U(\xi)} |\nabla u^+|^2\right)^{1/2} \leq \beta(k)$$

$$\int_{U(\xi)} u^+\tilde{\zeta} \leq 4ka\sqrt{\pi}||\omega_0||_2 \left(\int_{U(\xi)} |\nabla u^+|^2\right)^{1/2} \leq \beta(k)$$

for all $\xi \geq \xi_1, c \geq C_2(k) = \max\{C_1, 2ka\sqrt{\pi}\}$. From (2.5.36) we have

$$F(\tilde{\zeta}) \leq \frac{1}{2}(\beta(k) + 1). \tag{2.5.40}$$

We observe that

$$\begin{aligned} \gamma &= 2\tilde{\Psi}_\lambda(\tilde{\zeta}) - 2F(\tilde{\zeta}) + \int_\Omega \eta_c\tilde{\zeta} \\ &\geq 2\tilde{\Psi}_\lambda(\tilde{\zeta}) - 2F(\tilde{\zeta}). \end{aligned} \tag{2.5.41}$$

By Lemma 2.5.1 and (2.5.40)

$$\gamma \geq \frac{1}{2\pi} \log \frac{c}{2a} - \tilde{\beta}(k) \tag{2.5.42}$$

for all $\xi \geq \xi_1, c \geq C_2(k)$ where $\tilde{\beta}(k)$ is a constant independent of c.

By (2.5.35) and (2.5.42)

$$T_c\tilde{\zeta}(\mathbf{x}) - \eta_c(\mathbf{x}) > \frac{1}{2\pi} \log \frac{c}{2a} - \tilde{\beta}(k)$$

for almost all $\mathbf{x} \in A$. But $T_c\tilde{\zeta} \leq T_0\tilde{\zeta}$ almost everywhere hence

$$\int_{U(\xi)} \frac{1}{2\pi} \log \left(\frac{|\mathbf{x} - \bar{\mathbf{y}}|}{|\mathbf{x} - \mathbf{y}|} \right) \tilde{\zeta}(\mathbf{y}) d\mathbf{y} - \eta_c(\mathbf{x}) = T_0\tilde{\zeta}(\mathbf{x}) - \eta_c(\mathbf{x}) \geq \frac{1}{2\pi} \log \frac{c}{2a} - \tilde{\beta}(k)$$

for almost all $\mathbf{x} \in A$.

Let $R \geq 1$. For $\mathbf{x} \in A$ let $B = \{\mathbf{y} \in U(\xi) : |\mathbf{y} - \mathbf{x}| < Ra/c\}$. Then

$$\int_B \log \left(\frac{2a|\mathbf{x} - \bar{\mathbf{y}}|}{c|\mathbf{x} - \mathbf{y}|} \right) \tilde{\zeta}(\mathbf{y}) d\mathbf{y} + \int_{U(\xi)\backslash B} \log \left(\frac{2a|\mathbf{x} - \bar{\mathbf{y}}|}{c|\mathbf{x} - \mathbf{y}|} \right) \tilde{\zeta}(\mathbf{y}) d\mathbf{y} \geq 2\pi(\eta_c(\mathbf{x}) - \tilde{\beta}(k)) \tag{2.5.43}$$

for almost all $\mathbf{x} \in A$.

It is shown in Appendix A that there exist positive constants $M_1, M_2, M_3 > 0$ independent of c and ξ such that

$$\int_B \log \left(\frac{2a|\mathbf{x} - \bar{\mathbf{y}}|}{c|\mathbf{x} - \mathbf{y}|} \right) \tilde{\zeta}(\mathbf{y}) d\mathbf{y} \leq \begin{cases} (M_1 + M_2|\log x_2|)||\omega_0||_p & \text{if } x_2 \geq a, \\ M_3||\omega_0||_p & \text{if } 0 < x_2 \leq a. \end{cases} \tag{2.5.44}$$

Note that for $\mathbf{y} \in U(\xi)\backslash B$ we have $|\mathbf{x} - \mathbf{y}| \geq Ra/c$ and $|\mathbf{x} - \bar{\mathbf{y}}| \leq 4\xi$ hence

$$\int_{U(\xi)\backslash B} \log \left(\frac{2a|\mathbf{x} - \bar{\mathbf{y}}|}{c|\mathbf{x} - \mathbf{y}|} \right) \tilde{\zeta}(\mathbf{y}) d\mathbf{y} \leq \log \left(\frac{8\xi}{R} \right) \int_{U(\xi)\backslash B} \tilde{\zeta}(\mathbf{y}) d\mathbf{y}.$$

Then rearranging (2.5.43) and applying the estimate (2.5.44) we obtain

$$\log \left(\frac{R}{8\xi} \right) \int_{U(\xi)\backslash B} \tilde{\zeta}(\mathbf{y}) d\mathbf{y} \leq \begin{cases} (M_1 + M_2|\log x_2|)||\omega_0||_p + 2\pi(\tilde{\beta}(k) - \eta_c(\mathbf{x})) & \text{if } x_2 \geq a, \\ M_3||\omega_0||_p + 2\pi(\tilde{\beta}(k) - \eta_c(\mathbf{x})) & \text{if } 0 < x_2 \leq a \end{cases} \tag{2.5.45}$$

hence

$$\log\left(\frac{R}{8\xi}\right)\int_{U(\xi)\backslash B}\tilde{\zeta}(\mathbf{y})d\mathbf{y} < M(k). \tag{2.5.46}$$

Let $R(k,\xi) = 8\xi e^{2M(k)}$ and suppose ess diam$(A) > 2R(k,\xi)a/c$. By (2.5.46)

$$\int_{U(\xi)\backslash B}\tilde{\zeta}(\mathbf{y})d\mathbf{y} < \frac{1}{2} \tag{2.5.47}$$

for almost all $x \in A$. Then $A = A' \cup N$ where diam$(A') > 2R(k,\xi)a/c$, $\mu_2(N) = 0$ and (2.5.47) holds for all $x \in A'$. Choose $\mathbf{x}', \mathbf{x}'' \in A'$ such that $B_{R(k,\xi)a/c}(\mathbf{x}') \cap B_{R(k,\xi)a/c}(\mathbf{x}'') = \emptyset$. Then

$$1 \leq \int_{U(\xi)\backslash B_{R(k,\xi)a/c}(\mathbf{x}')}\tilde{\zeta}(\mathbf{y})d\mathbf{y} + \int_{U(\xi)\backslash B_{R(k,\xi)a/c}(\mathbf{x}'')}\tilde{\zeta}(\mathbf{y})d\mathbf{y} < \frac{1}{2} + \frac{1}{2} = 1$$

which is a contradiction. □

For $t \in \mathbb{R}$ and $\xi, \xi' > 0$ define

$$\Omega_c(t,\xi) = \{\mathbf{x} \in \Omega_c | |\mathbf{x} - (t,0)| < \xi\}.$$

and, if $|t| < \xi$, define

$$\Omega_c(t,\xi',\xi) = \Omega_c(t,\xi') \cap \Omega_c(\xi).$$

Let $\mathcal{F}_c(t,\xi',\xi)$ be the set of functions in $\mathcal{F}_c$ that vanish outside $\Omega_c(t,\xi',\xi)$. Let $\tilde{\zeta}_{c,t,\xi',\xi}$ denote a maximiser of $\tilde{\Psi}_\lambda$ relative to $\mathcal{F}_c(t,\xi',\xi)$ and let

$$A_{c,t,\xi',\xi} = \{\mathbf{x} \in \Omega_c(t,\xi',\xi) | \tilde{\zeta}_{c,t,\xi',\xi}(\mathbf{x}) > 0\}.$$

From Lemma 2.5.2 we immediately obtain the following result.

LEMMA 2.5.3 *Let $0 < \xi' \leq \xi$ and $|t| < \xi$ with either*

$$\{\mathbf{x} \in \Omega_c(t,\xi') | x_1 < t\} \subset \Omega_c(\xi) \tag{2.5.48}$$

or $$\{\mathbf{x} \in \Omega_c(t,\xi') | x_1 > t\} \subset \Omega_c(\xi). \tag{2.5.49}$$

Then there exist ξ_0, C_0 such that

$$\textit{ess diam}(A_{c,t,\xi',\xi}) \leq \frac{R(\xi')a}{c} \leq \frac{\sigma\xi' a}{c} \qquad \forall \xi \geq \xi' \geq \xi_0, c \geq C_0$$

where $\sigma > 0$ is a constant.

Proof We observe $\Omega_c(t,\xi',\xi) = \tilde{U}(t,\xi',\xi) \cap \Omega_c$ where $\tilde{U}(t,\xi',\xi) = B_{\xi'}((t,0)) \cap B_\xi(0)$ is a convex subset of Π and there exists ξ_0 such that for all $\xi \geq \xi' \geq \xi_0$ and $|t| < \xi$ satisfying (2.5.48) or (2.5.49), a fixed cone may be used to determine the cone property for $\tilde{U}(t,\xi',\xi)$. Furthermore

$$\begin{aligned} 2\xi' \geq \mu_1(\partial\tilde{U}(t,\xi',\xi) \cap \partial\Pi) &\geq \xi', \\ \mu_1(\partial\tilde{U}(t,\xi',\xi) \cap \partial\Pi) &\geq \frac{1}{2}\text{diam}(\tilde{U}(t,\xi',\xi)) \end{aligned}$$

and

$$B_{1/4\pi}((s,1/4\pi)) \subset \Omega_c(t,\xi',\xi)$$

for some $s \in \partial\tilde{U}(t,\xi',\xi) \cap \partial\Pi$. The result now follows from Lemma 2.5.2. □

Let

$$A_{c,\xi} = \{\tilde{\zeta}_{c,\xi} > 0\}.$$

LEMMA 2.5.4 *There exist ξ_0, C_0 such that for all $\xi \geq \xi_0, c \geq C_0$,*

$$A_{c,\xi} \subset \Omega_c(t_{c,\xi},\xi_0)$$

except for a set of measure zero, for some $t_{c,\xi} \in \mathbb{R}$.

Proof By Lemma 2.5.3 there exist ξ_1, C_1, σ such that

$$\text{ess diam}(A_{c,t,\xi',\xi}) \leq \frac{\sigma\xi' a}{c}$$

for all $\xi \geq \xi' \geq \xi_1, c \geq C_1$ and $|t| < \xi$ satisfying (2.5.48) or (2.5.49).

Fix $c > C_2 = \max\{C_1, 8\pi a, 4\pi(a+4), 4\sigma a\}$ and let

$d_\xi = \min\{\xi' : A_{c,\xi} \subset \Omega_c(t_{c,\xi},\xi')$ except for a set of measure zero, for some $t_{c,\xi} \in \mathbb{R}\}$.

For $\xi \geq \xi_1$ suppose $d_\xi \geq \xi_1$. Then $A_{c,\xi} \subseteq \Omega_c(t_{c,\xi}, d_\xi)$ except for a set of measure zero for some $t_{c,\xi}$ with $|t_{c,\xi}| < \xi$ and

$$\begin{aligned} &\{\mathbf{x} \in \Omega_c(t_{c,\xi}, d_\xi) | x_1 < t_{c,\xi}\} \subset \Omega_c(\xi) \\ \text{or} \quad &\{\mathbf{x} \in \Omega_c(t_{c,\xi}, d_\xi) | x_1 > t_{c,\xi}\} \subset \Omega_c(\xi). \end{aligned}$$

Also $\tilde{\zeta}_{c,\xi}$ maximises $\tilde{\Psi}_\lambda$ relative to $\mathcal{F}_c(t_{c,\xi}, d_\xi, \xi)$, and the conditions of Lemma (2.5.3) are satisfied. By the choice of C_2 we have ess $\operatorname{diam}(A_{c,t_{c,\xi},d_\xi,\xi}) \le d_\xi/4$, hence $A_{c,t_{c,\xi},d_\xi,\xi} = A' \cup N$ where $\operatorname{diam}(A') \le d_\xi/4$ and $\mu_2(N) = 0$. If there exists $\mathbf{x} \in A'$ with $x_2 \le d_\xi/2$ then it follows that there exist $t'_{c,\xi} \in \mathbb{R}$ and $d'_\xi < d_\xi$ such that $A_{c,\xi}$ is contained in $\Omega_c(t'_{c,\xi}, d'_\xi, \xi)$ except for a set of measure zero which is a contradiction. Therefore

$$A_{c,\xi} \subset \{\mathbf{x} \in \Omega_c(t_{c,\xi}, d_\xi, \xi) | x_2 > d_\xi/2\} \tag{2.5.50}$$

except for a set of measure zero.

Hence for almost all $\mathbf{x}, \mathbf{y} \in A_{c,\xi}$ we have $x_2, y_2 > d_\xi/2$ and

$$\begin{aligned}
&\frac{1}{2}h(c\mathbf{x}, c\mathbf{y}) - \frac{1}{4\pi}\log\frac{1}{c} + \eta_c(\mathbf{x}) \\
\ge\ & \frac{1}{4\pi}\log\frac{1}{|\mathbf{x}-\bar{\mathbf{y}}|} + x_2 - \frac{x_2}{c^2|\mathbf{x}|^2} \qquad \text{by (2.3.6) and (2.5.30)} \\
\ge\ & \frac{1}{4\pi}\log\frac{1}{2d_\xi} + \frac{d_\xi}{2} - \frac{2}{d_\xi} \\
\to\ & \infty \text{ as } d_\xi \to \infty.
\end{aligned}$$

Also, for all $\mathbf{x}, \mathbf{y} \in \operatorname{supp} \hat{\zeta}_c \subset B_{1/8\pi}(0, 1/4\pi)$

$$\begin{aligned}
&\frac{1}{2}h(c\mathbf{x}, c\mathbf{y}) - \frac{1}{4\pi}\log\frac{1}{c} + \eta_c(\mathbf{x}) \\
\le\ & \frac{1}{2}h_0(c\mathbf{x}, c\mathbf{y}) + \frac{1}{2}\hat{h}_1(c\mathbf{x}, c\mathbf{y}) - \frac{1}{4\pi}\log\frac{1}{c} + x_2 \\
\le\ & \frac{1}{4\pi}\log 4\pi + \frac{c^2 x_2 y_2}{2\pi(c^2|\mathbf{x}||\mathbf{y}| - 1)^2} + \frac{3}{8\pi} \quad \text{by (2.3.10)} \\
\le\ & \frac{1}{4\pi}\log 4\pi + \frac{2}{\pi} + \frac{3}{8\pi}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\tilde{\Psi}_\lambda(\hat{\zeta}_c) - \tilde{\Psi}_\lambda(\tilde{\zeta}_{c,\xi}) \\
=\ & \int_\Omega\int_\Omega \left(\frac{1}{4\pi}\log\frac{1}{|\mathbf{x}-\mathbf{y}|} - \left(\frac{1}{2}h(c\mathbf{x}, c\mathbf{y}) - \frac{1}{4\pi}\log\frac{1}{c} + \eta_c(\mathbf{x})\right)\right)\hat{\zeta}_c(\mathbf{x})\hat{\zeta}_c(\mathbf{y})
\end{aligned}$$

$$
\begin{aligned}
& -\int_\Omega\int_\Omega \left(\frac{1}{4\pi}\log\frac{1}{|\mathbf{x}-\mathbf{y}|} - \left(\frac{1}{2}h(c\mathbf{x},c\mathbf{y}) - \frac{1}{4\pi}\log\frac{1}{c} + \eta_c(\mathbf{x})\right)\right)\tilde{\zeta}_{c,\xi}(\mathbf{x})\tilde{\zeta}_{c,\xi}(\mathbf{y}) \\
\geq\ & \int_\Omega\int_\Omega \left(\frac{1}{2}h(c\mathbf{x},c\mathbf{y}) - \frac{1}{4\pi}\log\frac{1}{c} + \eta_c(\mathbf{x})\right)\tilde{\zeta}_{c,\xi}(\mathbf{x})\tilde{\zeta}_{c,\xi}(\mathbf{y}) \\
& -\int_\Omega\int_\Omega \left(\frac{1}{2}h(c\mathbf{x},c\mathbf{y}) - \frac{1}{4\pi}\log\frac{1}{c} + \eta_c(\mathbf{x})\right)\hat{\zeta}_c(\mathbf{x})\hat{\zeta}_c(\mathbf{y}) \\
>\ & 0
\end{aligned}
$$

if d_ξ is sufficiently large independently of c hence d_ξ is bounded. □

THEOREM 2.5.5 *There exist ξ_0, C_0 such that for all $\xi \geq \xi_0, c \geq C_0$,*

$$A_{c,\xi} \subset \Omega_c(\xi_0)$$

except for a set of measure zero.

Proof We use the asymptotic expansion for η

$$\eta(\mathbf{x}) = x_2 + \frac{2d_2x_2}{|\mathbf{x}|^2} + O(|\mathbf{x}|^{-2})$$

derived by Turkington [37, Lemma 5.1] where

$$d_2 = \frac{-1}{2\pi}\left(\int_\Omega |\nabla(\eta - x_2)|^2 + \frac{1}{2}\mu_2(D)\right).$$

We observe that $d_2 < 0$ if $D \neq \emptyset$. Choose constants $K_1, K_2 > 0$ with $K_2 > \max\{2, K_1/|d_2|\}$ such that

$$\left|\frac{1}{c}\eta(c\mathbf{x}) - x_2 - \frac{2d_2x_2}{c^2|\mathbf{x}|^2}\right| \leq \frac{K_1}{c^3|\mathbf{x}|^2} \text{ if } |\mathbf{x}| > \frac{K_2}{c}. \tag{2.5.51}$$

By Theorem 2.5.4 there exist ξ_1, C_1 such that for all $\xi \geq \xi_1$, $c \geq C_1$ we have

$$A_{c,\xi} \subset \Omega_c(t_{c,\xi}, \xi_1)$$

except for a set of measure zero for some $t_{c,\xi} \in \mathbb{R}$. We show $t_{c,\xi}$ is bounded for sufficiently large c and ξ.

We recall some results from Section 2.4. By Lemma 2.4.3, Theorem 2.4.4 and Lemma 2.4.8 there exist $\xi_2 \geq \xi_1, C_2 \geq C_1$ such that

$$S_{c,\xi_2} = \{\zeta^\sharp_{c,\xi_2} > 0\} \subset B_{1/8\pi}(\hat{\mathbf{X}}) \qquad \forall \xi \geq \xi_2, c \geq C_2$$

where $\zeta^\sharp_{c,\xi_2}$ is a Steiner-symmetric maximiser of E relative to $\mathcal{F}_c$.

Let $s = |d_2|^{-1/2} + 1/8\pi$ and, for $\xi \geq \xi_3 = \max\{2\xi_2, 2s\}, c \geq C_2$, let

$$\Gamma_c(\xi) = \{\mathbf{x} \in \Omega_c(\xi) | x_2 > K_2/c\}.$$

Define $\zeta^\sharp_{c,\xi_2,s}(x_1, x_2) = \zeta^\sharp_{c,\xi_2}(x_1 - s, x_2)$, and for $\mathbf{x} \in \Omega_c(\xi)$ define

$$\begin{aligned} Q(\mathbf{x}) &= \left(\frac{1}{2}T_c\zeta^\sharp_{c,\xi_2,s}(\mathbf{x}) - \eta_c(\mathbf{x})\right) - \left(\frac{1}{2}T_0\zeta^\sharp_{c,\xi_2,s}(\mathbf{x}) - x_2\right) \\ &= \frac{1}{2}\left(T_c\zeta^\sharp_{c,\xi_2,s}(\mathbf{x}) - T_0\zeta^\sharp_{c,\xi_2,s}(\mathbf{x})\right) - (\eta_c(\mathbf{x}) - x_2) \end{aligned}$$

where we note that $(T_c\zeta^\sharp_{c,\xi_2,s}(\mathbf{x}) - T_0\zeta^\sharp_{c,\xi_2,s}(\mathbf{x})) \leq 0$ and $(\eta_c(\mathbf{x}) - x_2) \leq 0$ for almost all $\mathbf{x} \in \Omega_c(\xi)$. Let $S_{c,\xi_2,s} = \{\zeta^\sharp_{c,\xi_2,s} > 0\}$. Then for $\mathbf{x} \in \Omega_c$ we have

$$\begin{aligned} |\frac{1}{2}(T_c - T_0)\zeta^\sharp_{c,\xi_2,s}(\mathbf{x})| &= \left|\int_{\Omega_c}\left(\frac{1}{4\pi}\log\frac{1}{c|\mathbf{x} - \bar{\mathbf{y}}|} - \frac{1}{2}h(c\mathbf{x}, c\mathbf{y})\right)\zeta^\sharp_{c,\xi_2,s}(\mathbf{y})d\mathbf{y}\right| \\ &\leq \int_{\Omega_c}|\frac{1}{2}\hat{h}(c\mathbf{x}, c\mathbf{y})|\zeta^\sharp_{c,\xi_2,s}(\mathbf{y})d\mathbf{y} \\ &\leq \sup_{\mathbf{y}\in S_{c,\xi_2,s}}|\frac{1}{2}\hat{h}_1(c\mathbf{x}, c\mathbf{y})| \\ &\leq \sup_{\mathbf{y}\in S_{c,\xi_2,s}}\frac{c^2x_2y_2}{2\pi(c^2|\mathbf{x}||\mathbf{y}| - 1)^2} \quad \text{by (2.3.10).} \end{aligned} \tag{2.5.52}$$

Let $\mathbf{x} \in \partial\Gamma_c(\xi)$ with $|\mathbf{x}| = \xi$. If $c \geq C_3 = \max\{C_2, |d_2|^{1/2}\}$ then for almost all $\mathbf{y} \in S_{c,\xi_2,s}$

$$\frac{1}{2}c^2|\mathbf{x}||\mathbf{y}| - 1 \geq \frac{1}{2}c^2|\mathbf{x}|\left(s - \frac{1}{8\pi}\right) - 1 > 0$$

hence

$$c^2|\mathbf{x}||\mathbf{y}| - 1 > \frac{1}{2}c^2|\mathbf{x}||\mathbf{y}|$$

and by (2.5.52)

$$|\frac{1}{2}(T_c - T_0)\zeta^\sharp_{c,\xi_2,s}(\mathbf{x})| \leq \frac{2x_2(3/8\pi)}{\pi c^2|\mathbf{x}|^2(s-1/8\pi)^2} \leq \frac{|d_2|x_2}{8c^2|\mathbf{x}|^2}.$$

But by (2.5.51)

$$|\eta_c(\mathbf{x}) - x_2| \geq \frac{2|d_2|x_2}{c^2|\mathbf{x}|^2} - \frac{K_1}{c^3|\mathbf{x}|^2} > \frac{|d_2|x_2}{c^2|\mathbf{x}|^2}$$

since

$$\frac{K_1}{c} < \frac{K_1}{c}\frac{cx_2}{K_2} < |d_2|x_2$$

by the definition of $\Gamma_c(\xi)$ and the choice of K_2. Hence

$$Q(\mathbf{x}) > \frac{|d_2|x_2}{2c^2|\mathbf{x}|^2} \quad \forall \mathbf{x} \in \partial\Gamma_c(\xi) \text{ with } |\mathbf{x}| = \xi. \tag{2.5.53}$$

Now consider $\mathbf{x} = (x_1, K_2/c) \in \partial\Gamma_c(\xi)$. For almost all $\mathbf{y} \in S_{c,\xi_2,s}$

$$\frac{1}{2}c^2|\mathbf{x}||\mathbf{y}| - 1 \geq \frac{1}{2}cK_2|d_2|^{-1/2} - 1 > 0$$

and by (2.5.52)

$$|\frac{1}{2}(T_c - T_0)\zeta^\sharp_{c,\xi_2,s}(\mathbf{x})| \leq \frac{2x_2(3/8\pi)}{\pi c^2|\mathbf{x}|^2(s-1/8\pi)^2} \leq \frac{|d_2|x_2}{8c^2|\mathbf{x}|^2}.$$

Also, since $|d_2|K_2 > K_1$ we have

$$|\eta_c(\mathbf{x}) - x_2| \geq \frac{2|d_2|x_2}{c^2|\mathbf{x}|^2} - \frac{K_1}{c^3|\mathbf{x}|^2} \geq \frac{|d_2|x_2}{c^2|\mathbf{x}|^2}.$$

Therefore

$$Q(\mathbf{x}) > \frac{|d_2|x_2}{2c^2|\mathbf{x}|^2} \tag{2.5.54}$$

for all $\mathbf{x} \in \partial\Gamma_c(\xi)$ with $x_2 = K_2/c$. Combining (2.5.53) and (2.5.54) yields

$$U(\mathbf{x}) := Q(\mathbf{x}) - \frac{|d_2|x_2}{2c^2|\mathbf{x}|^2} > 0 \quad \forall \mathbf{x} \in \partial\Gamma_c(\xi)$$

for all $c \geq C_3, \xi \geq \xi_3$. Also

$$-\Delta U = 0 \text{ almost everywhere in } \Gamma_c(\xi)$$

since $-\Delta Q = 0$ almost everywhere in Ω_c. By the maximum principle $U > 0$ almost everywhere in $\Gamma_c(\xi)$ hence

$$\frac{1}{2} T_c \zeta^\sharp_{c,\xi_2,s}(\mathbf{x}) - \eta_c(\mathbf{x}) \geq \frac{1}{2} T_0 \zeta^\sharp_{c,\xi_2,s}(\mathbf{x}) - x_2 + \frac{|d_2| x_2}{2c^2|\mathbf{x}|^2}$$

for almost all $\mathbf{x} \in \Gamma_c(\xi)$ and

$$\int_{\Omega_c} \left(\frac{1}{2} T_c \zeta^\sharp_{c,\xi_2,s} - \eta_c \right) \zeta^\sharp_{c,\xi_2,s} > \int_{\Omega_c} \left(\frac{1}{2} T_0 \zeta^\sharp_{c,\xi_2,s} - x_2 \right) \zeta^\sharp_{c,\xi_2,s} + \int_{\Omega_c} \frac{|d_2| x_2}{2c^2|\mathbf{x}|^2} \zeta^\sharp_{c,\xi_2,s}.$$

Since $x_2 > 1/8\pi$ and $|\mathbf{x}| < s + 3/8\pi$ for almost all $\mathbf{x} \in S_{c,\xi_2,s}$

$$\begin{aligned} \tilde{\Psi}_\lambda(\tilde{\zeta}_{c,\xi}) \geq \tilde{\Psi}_\lambda(\zeta^\sharp_{c,\xi_2,s}) &> E(\zeta^\sharp_{c,\xi_2,s}) + \frac{|d_2|}{16\pi(s+3/8\pi)^2 c^2} \\ &= \sup_{v \in \mathcal{F}_c} E(v) + \frac{|d_2|}{16\pi(s+3/8\pi)^2 c^2} \end{aligned} \tag{2.5.55}$$

for all $\xi \geq \xi_3, c \geq C_3$.

Fix $c \geq C_3$ and let $\xi \geq 3\xi_3$. Suppose $|t_{c,\xi}| > 2\xi_3$. Then by Lemma 2.5.4 we have for almost all $\mathbf{x}, \mathbf{y} \in \operatorname{supp} \tilde{\zeta}_{c,\xi}$

$$\frac{1}{2} c^2 |\mathbf{x}||\mathbf{y}| - 1 > \frac{1}{2} c^2 (|t_{c,\xi}| - \xi_3)^2 - 1 > \frac{1}{2} c^2 \xi_3^2 - 1 > 0$$

and therefore

$$c^2 |\mathbf{x}||\mathbf{y}| - 1 > \frac{1}{2} c^2 |\mathbf{x}||\mathbf{y}| > \frac{1}{2} c^2 \left(\frac{|t_{c,\xi}|}{2} \right)^2. \tag{2.5.56}$$

By (2.5.55)

$$\begin{aligned} \frac{|d_2|}{16\pi(s+3/8\pi)^2 c^2} &\leq \tilde{\Psi}_\lambda(\tilde{\zeta}_{c,\xi}) - E(\zeta^\sharp_{c,\xi_2,s}) \\ &\leq \tilde{\Psi}_\lambda(\tilde{\zeta}_{c,\xi}) - E(\tilde{\zeta}_{c,\xi}) \end{aligned}$$

and

$$
\begin{aligned}
& |\tilde{\Psi}_\lambda(\tilde{\zeta}_{c,\xi}) - E(\tilde{\zeta}_{c,\xi})| \\
= \; & \left| \int_{\Omega_c}\int_{\Omega_c} \left(\frac{1}{2}h_0(c\mathbf{x}, c\mathbf{y}) + x_2\right) \tilde{\zeta}_{c,\xi}(\mathbf{x})\tilde{\zeta}_{c,\xi}(\mathbf{y}) - \int_{\Omega_c}\int_{\Omega_c} \left(\frac{1}{2}h(c\mathbf{x}, c\mathbf{y}) + \eta_c(\mathbf{x})\right) \tilde{\zeta}_{c,\xi}(\mathbf{x})\tilde{\zeta}_{c,\xi}(\mathbf{y}) \right| \\
\le \; & \sup_{\mathbf{x},\mathbf{y}\in A_{c,\xi}} |\frac{1}{2}\hat{h}_1(c\mathbf{x}, c\mathbf{y})| + \sup_{\mathbf{x}\in A_{c,\xi}} |\eta_c(\mathbf{x}) - x_2| \\
\le \; & \sup_{\mathbf{x},\mathbf{y}\in A_{c,\xi}} \frac{c^2 x_2 y_2}{2\pi(c^2|\mathbf{x}||\mathbf{y}| - 1)^2} + \sup_{\mathbf{x}\in A_{c,\xi}} \frac{x_2}{c^2|\mathbf{x}|^2} \qquad \text{by (2.3.10) and (2.5.30)} \\
\le \; & \frac{3}{c^2|t_{c,\xi}|^2} + \frac{2}{c^2|t_{c,\xi}|} \quad \text{by (2.5.56)} \\
\le \; & \frac{3}{c^2|t_{c,\xi}|}.
\end{aligned}
$$

Hence

$$
\frac{|d_2|}{16\pi(s + 3/8\pi)^2 c^2} \le \frac{3}{c^2|t_{c,\xi}|}
$$

and since $s = |d_2|^{-1/2} + 1/8\pi$

$$
|t_{c,\xi}| \le \frac{48\pi(|d_2|^{-1/2} + 1/2\pi)^2}{|d_2|}.
$$

Thus $t_{c,\xi}$ is bounded if $c \ge C_3$ and $\xi > 3\xi_3$. This completes the proof. □

With the notation introduced in 2.5.1 we have

THEOREM 2.5.6 *There exists $\lambda_0 > 0$ such that if $0 < \lambda < \lambda_0$ then Ψ_λ attains a maximum relative to $\mathcal{F}$. If ω_λ is a maximiser and $\psi_\lambda = T\omega_\lambda$, then*

$$
-\Delta\psi_\lambda = \phi \circ (\psi_\lambda - \lambda\eta)
$$

almost everywhere in Ω for some increasing function ϕ.

Proof Recall

$$
\tilde{\omega}_{\lambda,\xi/\lambda}(\mathbf{x}) = \frac{1}{c^2}\tilde{\zeta}_{c,\xi}\left(\frac{\mathbf{x}}{c}\right)
$$

and $\tilde{\omega}_{\lambda,\xi/\lambda}$ maximises Ψ_λ relative to $\mathcal{F}(\xi/\lambda)$. By Theorem 2.5.5, for $\xi \ge \xi_0$, $0 < \lambda < \lambda_0 = 1/C_0$ the support of $\tilde{\omega}_{\lambda,\xi/\lambda}$ is contained in $\Omega(\xi_0/\lambda)$ except for a set of measure zero, hence $\tilde{\omega}_{\lambda,\xi_0/\lambda}$ maximises Ψ_λ relative to $\mathcal{F}$.

Write $\psi_\lambda = T\tilde{\omega}_{\lambda,\xi_0/\lambda}$. Then $\tilde{\omega}_{\lambda,\xi_0/\lambda} = \phi_\lambda \circ (\psi_\lambda - \lambda\eta)$ almost everywhere in $\Omega(\xi_0/\lambda)$ for some increasing function ϕ_λ for which we can assume that $\phi_\lambda(s) \geq 0$ for all $s \in \text{dom } \phi_\lambda$. Since $\tilde{\omega}_{\lambda,\xi_0/\lambda}$ is an increasing function of $(\psi_\lambda - \lambda\eta)$ almost everywhere in $\Omega(\xi/\lambda)$ for all $\xi \geq \xi_0$, there exists γ_λ such that $\psi_\lambda - \lambda\eta \geq \gamma_\lambda$ almost everywhere on $\{\tilde{\omega}_{\lambda,\xi/\lambda} > 0\}$ and $\psi_\lambda - \lambda\eta < \gamma_\lambda$ almost everywhere in $\Omega\backslash\Omega(\xi_0/\lambda)$. Define

$$\phi(s) = \begin{cases} \phi_\lambda(s) & \text{if } s \in \text{dom } \phi_\lambda,\ s \geq \gamma_\lambda \\ 0 & \text{if } s < \gamma_\lambda \end{cases}$$

Then ϕ is increasing and $\tilde{\omega}_{\lambda,\xi_0/\lambda} = \phi \circ (\psi_\lambda - \lambda\eta)$ almost everywhere in Ω. □

2.5.3 Asymptotic properties of the maximisers

Henceforth $\tilde{\zeta}_c$ will denote a maximiser of $\tilde{\Psi}_\lambda$ relative to $\mathcal{F}_c$ and $\tilde{\omega}_\lambda$ the corresponding maximiser of Ψ_λ relative to $\mathcal{F}$. Let $R = R(\xi_0)$ be as in Lemma 2.5.3 (taking $t = 0$ and $\xi' = \xi$) and let $\tilde{\mathbf{X}}_c = (\tilde{X}_{c1}, \tilde{X}_{c2}) = \int_{\Omega_c} \mathbf{x}\tilde{\zeta}_c(\mathbf{x})d\mathbf{x}$ denote the centre of vorticity.

We shall use the terminology introduced in 2.4.3 which enables us to work with "sequences" of maximisers.

We state a result which is analogous to Lemma 2.4.7 and immediate using the same methods.

LEMMA 2.5.7 *For any sequence $\tilde{\mathbf{X}}_c$,*

$$c\tilde{X}_{c2} \to \infty \qquad \text{as } c \to \infty \tag{2.5.57}$$

and, in particular, there exists C_0 such that for $c \geq C_0$

$$\text{supp } \tilde{\zeta}_c \subset \{\mathbf{x} \in \Omega_c | x_2 > 1/c\}$$

except for a set of measure zero.

LEMMA 2.5.8 *For any sequence $\tilde{\mathbf{X}}_c$,*

$$\left|\tilde{X}_{c2} - \frac{1}{4\pi}\right| \to 0 \quad \text{as } c \to \infty.$$

Proof Let $\hat{\zeta}_c$ be the circular symmetric decreasing rearrangement of $\tilde{\zeta}_c$ relative to the point $\hat{\mathbf{X}} = (0, 1/4\pi)$. Then

$$\int_{\Omega_c}\int_{\Omega_c}\left(\frac{1}{2}h(c\mathbf{x}, c\mathbf{y}) - \frac{1}{4\pi}\log\frac{1}{c} + \frac{1}{c}\eta(c\mathbf{x})\right)\tilde{\zeta}_c(\mathbf{x})\tilde{\zeta}_c(\mathbf{y})d\mathbf{x}d\mathbf{y} \quad (2.5.58)$$

$$= \int_{\Omega_c}\int_{\Omega_c}\frac{1}{4\pi}\log\frac{1}{|\mathbf{x}-\mathbf{y}|}\tilde{\zeta}_c(\mathbf{x})\tilde{\zeta}_c(\mathbf{y})d\mathbf{x}d\mathbf{y} - \tilde{\Psi}_\lambda(\tilde{\zeta}_c)$$

$$\leq \int_{\Omega_c}\int_{\Omega_c}\frac{1}{4\pi}\log\frac{1}{|\mathbf{x}-\mathbf{y}|}\hat{\zeta}_c(\mathbf{x})\hat{\zeta}_c(\mathbf{y})d\mathbf{x}d\mathbf{y} - \tilde{\Psi}_\lambda(\hat{\zeta}_c)$$

$$= \int_{\Omega_c}\int_{\Omega_c}\left(\frac{1}{2}h(c\mathbf{x}, c\mathbf{y}) - \frac{1}{4\pi}\log\frac{1}{c} + \frac{1}{c}\eta(c\mathbf{x})\right)\hat{\zeta}_c(\mathbf{x})\hat{\zeta}_c(\mathbf{y})d\mathbf{x}d\mathbf{y} \quad (2.5.59)$$

We determine the limiting values of (2.5.58) and (2.5.59) as $c \to \infty$.

$$\left|\int_{\Omega_c}\int_{\Omega_c}\left(\frac{1}{2}h(c\mathbf{x}, c\mathbf{y}) - \frac{1}{4\pi}\log\frac{1}{c}\right)\hat{\zeta}_c(\mathbf{x})\hat{\zeta}_c(\mathbf{y})d\mathbf{x}d\mathbf{y} - \frac{1}{2}h_0(\hat{\mathbf{X}}, \hat{\mathbf{X}})\right|$$

$$\leq \sup_{\mathbf{x},\mathbf{y}\in B_{a/c}(\hat{\mathbf{X}})}\left|\frac{1}{2}h(c\mathbf{x}, c\mathbf{y}) - \frac{1}{4\pi}\log\frac{1}{c} - \frac{1}{4\pi}\log\frac{1}{2\hat{X}_2}\right|$$

$$\leq \sup_{\mathbf{x},\mathbf{y}\in B_{a/c}(\hat{\mathbf{X}})}\left|\frac{1}{2}\hat{h}_1(c\mathbf{x}, c\mathbf{y})\right| + \left|\frac{1}{4\pi}\log\frac{1}{c|\mathbf{x}-\bar{\mathbf{y}}|} - \frac{1}{4\pi}\log\frac{1}{2c\hat{X}_2}\right|$$

$$\leq \sup_{\mathbf{x},\mathbf{y}\in B_{a/c}(\hat{\mathbf{X}})}\left|\frac{c^2x_2y_2}{2\pi(c^2|\mathbf{x}||\mathbf{y}|-1)^2}\right| + \sup_{\mathbf{x},\mathbf{y}\in B_{a/c}(\hat{\mathbf{X}})}\left|\frac{1}{4\pi}\log\frac{2\hat{X}_2}{|\mathbf{x}-\bar{\mathbf{y}}|}\right| \quad \text{by (2.3.10)}$$

$$\to 0 \quad \text{as } c \to \infty \quad (2.5.60)$$

Also

$$\sup_{\mathbf{x}\in B_{a/c}(\hat{\mathbf{X}})}\left|\frac{1}{c}\eta(c\mathbf{x}) - \hat{X}_2\right| \leq \frac{a}{c} + \sup_{\mathbf{x}\in B_{a/c}(\hat{\mathbf{X}})}\frac{x_2}{c^2|\mathbf{x}|^2}$$

$$\leq \frac{a}{c} + \frac{1}{c(\frac{c}{4\pi} - a)}$$

$$\to 0 \text{ as } c \to \infty. \quad (2.5.61)$$

Note that

$$\left|\int_{\Omega_c}\int_{\Omega_c}\frac{1}{2}h(c\mathbf{x}, c\mathbf{y})\tilde{\zeta}_c(\mathbf{x})\tilde{\zeta}_c(\mathbf{y})d\mathbf{x}d\mathbf{y} - \frac{1}{2}h(c\tilde{\mathbf{X}}_c, c\tilde{\mathbf{X}}_c)\right|$$

$$\leq \sup_{\mathbf{x},\mathbf{y}\in B_{Ra/c}(\tilde{\mathbf{X}}_c)} \left|\frac{1}{2}h(c\mathbf{x}, c\mathbf{y}) - \frac{1}{2}h(c\tilde{\mathbf{X}}_c, c\tilde{\mathbf{X}}_c)\right| \tag{2.5.62}$$

and

$$\begin{aligned}
& h_0(c\mathbf{x}, c\mathbf{y}) - \hat{h}_1(c\tilde{\mathbf{X}}_c, c\tilde{\mathbf{X}}_c) - h_0(c\tilde{\mathbf{X}}_c, c\tilde{\mathbf{X}}_c) \\
= \; & h_0(c\mathbf{x}, c\mathbf{y}) - h_1(c\tilde{\mathbf{X}}_c, c\tilde{\mathbf{X}}_c) \\
\leq \; & h(c\mathbf{x}, c\mathbf{y}) - h(c\tilde{\mathbf{X}}_c, c\tilde{\mathbf{X}}_c) \\
\leq \; & h_1(c\mathbf{x}, c\mathbf{y}) - h_0(c\tilde{\mathbf{X}}_c, c\tilde{\mathbf{X}}_c) \\
= \; & \hat{h}_1(c\mathbf{x}, c\mathbf{y}) + h_0(c\mathbf{x}, c\mathbf{y}) - h_0(c\tilde{\mathbf{X}}_c, c\tilde{\mathbf{X}}_c)
\end{aligned} \tag{2.5.63}$$

for all $\mathbf{x}, \mathbf{y} \in B_{Ra/c}(\tilde{\mathbf{X}}_c)$ for c sufficiently large. But

$$\begin{aligned}
\sup_{\mathbf{x},\mathbf{y}\in B_{Ra/c}(\tilde{\mathbf{X}}_c)} \left|\frac{1}{2}\hat{h}_1(c\mathbf{x}, c\mathbf{y})\right| &\leq \sup_{\mathbf{x},\mathbf{y}\in B_{Ra/c}(\tilde{\mathbf{X}}_c)} \left|\frac{c^2 x_2 y_2}{2\pi(c^2|\mathbf{x}||\mathbf{y}| - 1)^2}\right| \\
&\leq \frac{(c|\tilde{\mathbf{X}}_c| + Ra)^2}{2\pi((c|\tilde{\mathbf{X}}_c| - Ra)^2 - 1)^2} \\
&\to 0 \quad \text{as } c \to \infty
\end{aligned} \tag{2.5.64}$$

and

$$\begin{aligned}
\sup_{\mathbf{x},\mathbf{y}\in B_{Ra/c}(\tilde{\mathbf{X}}_c)} \left|\frac{1}{2}h_0(c\mathbf{x}, c\mathbf{y}) - \frac{1}{2}h_0(c\tilde{\mathbf{X}}_c, c\tilde{\mathbf{X}}_c)\right| &= \sup_{\mathbf{x},\mathbf{y}\in B_{Ra/c}(\tilde{\mathbf{X}}_c)} \left|\frac{1}{4\pi}\log\frac{2\tilde{X}_{c2}}{|\mathbf{x} - \bar{\mathbf{y}}|}\right| \\
&\to 0 \quad \text{as } c \to \infty
\end{aligned} \tag{2.5.65}$$

since $c\tilde{X}_{c2} \to \infty$ as $c \to \infty$ by Lemma 2.5.7. Hence

$$\left|\int_{\Omega_c}\int_{\Omega_c} \frac{1}{2}h(c\mathbf{x}, c\mathbf{y})\tilde{\zeta}_c(\mathbf{x})\tilde{\zeta}_c(\mathbf{y})d\mathbf{x}d\mathbf{y} - \frac{1}{2}h(c\tilde{\mathbf{X}}_c, c\tilde{\mathbf{X}}_c)\right| \to 0 \quad \text{as } c \to \infty. \tag{2.5.66}$$

Finally

$$\begin{aligned}
\left|\int_{\Omega_c} \frac{1}{c}\eta(c\mathbf{x})\tilde{\zeta}_c(\mathbf{x})d\mathbf{x} - \tilde{X}_{c2}\right| &\leq \sup_{\mathbf{x}\in B_{Ra/c}(\tilde{\mathbf{X}}_c)} \left|\frac{1}{c}\eta(c\mathbf{x}) - \tilde{X}_{c2}\right| \\
&\leq \frac{Ra}{c} + \sup_{\mathbf{x}\in B_{Ra/c}(\tilde{\mathbf{X}}_c)} \frac{x_2}{c^2|\mathbf{x}|^2}
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{Ra}{c} + \frac{1}{c(c|\tilde{\mathbf{X}}_c| - Ra)} \\
&\to 0 \text{ as } c \to \infty. \qquad (2.5.67)
\end{aligned}
$$

Letting $c \to \infty$ in (2.5.58) and (2.5.59) using the estimates (2.5.60), (2.5.61), (2.5.66) and (2.5.67)

$$\frac{1}{2}h(c\tilde{\mathbf{X}}_c, c\tilde{\mathbf{X}}_c) - \frac{1}{4\pi}\log\frac{1}{c} + \tilde{X}_{c2} \leq \frac{1}{2}h_0(\hat{\mathbf{X}}, \hat{\mathbf{X}}) + \hat{X}_2 + o(1) \quad \text{as } c \to \infty.$$

Hence

$$\frac{1}{2}h_0(\tilde{\mathbf{X}}_c, \tilde{\mathbf{X}}_c) + \tilde{X}_{c2} \leq \frac{1}{2}h_0(\hat{\mathbf{X}}, \hat{\mathbf{X}}) + \hat{X}_2 + o(1) \quad \text{as } c \to \infty$$

which we rewrite as

$$H_0(\tilde{\mathbf{X}}_c) \leq H_0(\hat{\mathbf{X}}) + o(1) \qquad \text{as } c \to \infty.$$

For $z > 0$ define $\tilde{H}_0(z) = H_0((0, z))$. Then $\tilde{X}_{c2}$ forms a minimising sequence for $\tilde{H}_0$ and $\tilde{H}_0(z)$ is minimised when $z = 1/4\pi$. It follows that $\tilde{X}_{c2} \to 1/4\pi$ as $c \to \infty$. □

For a bounded domain $U \subset \mathbb{R}^2$ define a functional $\mathcal{I} : L^p(U) \to \mathbb{R}$ by

$$\mathcal{I}(w) = \frac{1}{4\pi}\int_U \int_U \log\frac{1}{|\mathbf{x} - \mathbf{y}|}w(\mathbf{x})w(\mathbf{y})d\mathbf{x}d\mathbf{y} \quad \forall w \in L^p(U)$$

and let

$$\mathcal{K}w(\mathbf{x}) = \frac{1}{2\pi}\int_U \log\frac{1}{|\mathbf{x} - \mathbf{y}|}w(\mathbf{y})d\mathbf{y}.$$

LEMMA 2.5.9 *Let $U \subset \mathbb{R}^2$ be a bounded domain with the cone property. Then $\mathcal{I}$ is weakly sequentially continuous and ,furthermore, $\mathcal{I}$ is Gateaux differentiable with $d\mathcal{I}[w] = Kw$.*

Proof It is shown in Appendix B that $\mathcal{K} : L^p(U) \to W^{2,p}(U)$ is bounded and, since the embedding $W^{2,p}(U) \to L^q(U)$ is compact, it follows that $\mathcal{K} : L^p(U) \to L^q(U)$ is compact and therefore $\mathcal{I}$ is weakly sequentially continuous.

For $w, u \in L^p(U)$ and $\alpha \in \mathbb{R}$

$$\frac{\mathcal{I}(w + \alpha u) - \mathcal{I}(w)}{\alpha} = \int_U u\mathcal{K}w + \alpha\mathcal{I}(u) \qquad (2.5.68)$$

hence $\mathcal{I}$ is Gateaux differentiable on $L^p(U)$ with $d\mathcal{I}[w] = \mathcal{K}w$. $\square$

Define

$$\tilde{f}_\lambda(\mathbf{x}) = \frac{1}{c^2}\tilde{\zeta}_c\left(\tilde{\mathbf{X}}_c + \frac{\mathbf{x}}{c}\right), \quad \mathbf{x} \in B_{Ra}(0). \tag{2.5.69}$$

Let $\hat{\omega}$ be the circularly symmetric decreasing rearrangement of ω_0 relative to the origin.

THEOREM 2.5.10 *Let $\tilde{\zeta}_c$ be a sequence of maximisers of $\tilde{\Psi}_\lambda$ relative to $\mathcal{F}_c$. Then $\tilde{f}_\lambda \to \hat{\omega}$ in $L^p(B_{Ra}(0))$ as $\lambda \to 0$.*

Proof Let $\mathcal{S}(\omega_0)$ denote the set of rearrangements of ω_0 in $L^p(B_{Ra}(0))$. Let

$$\mathcal{A} = \{v \in L^p(B_{Ra}(0)) | \int_{B_{Ra}(0)} v = 1\}.$$

Note that $\mathcal{A}$ is a convex subset of $L^p(B_{Ra}(0))$. For $v \in \mathcal{A}$ define

$$v_c(\mathbf{x}) = \begin{cases} c^2 v(c(\mathbf{x} - \tilde{\mathbf{X}}_c)) & \mathbf{x} \in B_{Ra/c}(\tilde{\mathbf{X}}_c) \\ 0 & \mathbf{x} \in \Omega_c(\xi_0)\backslash B_{Ra/c}(\tilde{\mathbf{X}}_c) \end{cases}$$

Then

$$\begin{aligned}
\tilde{\Psi}_\lambda(v_c) &= \int_{\Omega_c}\int_{\Omega_c} \frac{1}{4\pi}\log\frac{1}{|\mathbf{x}-\mathbf{y}|} v_c(\mathbf{x})v_c(\mathbf{y}) \\
&\quad - \int_{\Omega_c}\int_{\Omega_c}\left(\frac{1}{2}h(c\mathbf{x}, c\mathbf{y}) - \frac{1}{4\pi}\log\frac{1}{c} + \frac{1}{c}\eta(c\mathbf{x})\right) v_c(\mathbf{x})v_c(\mathbf{y}) \\
&= \int_{B_{Ra/c}(\tilde{\mathbf{X}}_c)}\int_{B_{Ra/c}(\tilde{\mathbf{X}}_c)} \frac{1}{4\pi}\log\frac{1}{|\mathbf{x}-\mathbf{y}|} c^2 v(c(\mathbf{x}-\tilde{\mathbf{X}}_c)) c^2 v(c(\mathbf{y}-\tilde{\mathbf{X}}_c)) \\
&\quad - \int_{\Omega_c}\int_{\Omega_c}\left(\frac{1}{2}h(c\mathbf{x}, c\mathbf{y}) - \frac{1}{4\pi}\log\frac{1}{c} + \frac{1}{c}\eta(c\mathbf{x})\right) v_c(\mathbf{x})v_c(\mathbf{y}) \\
&= \int_{B_{Ra}(0)}\int_{B_{Ra}(0)} \frac{1}{4\pi}\log\frac{c}{|\mathbf{x}-\mathbf{y}|} v(\mathbf{x})v(\mathbf{y}) \\
&\quad - \int_{\Omega_c}\int_{\Omega_c}\left(\frac{1}{2}h(c\mathbf{x}, c\mathbf{y}) - \frac{1}{4\pi}\log\frac{1}{c} + \frac{1}{c}\eta(c\mathbf{x})\right) v_c(\mathbf{x})v_c(\mathbf{y})
\end{aligned}$$

which we rewrite as

$$\tilde{\Psi}_\lambda(v_c) = \frac{1}{4\pi}\log c + \mathcal{I}(v) - \int_{\Omega_c}\int_{\Omega_c}\left(\frac{1}{2}h(c\mathbf{x}, c\mathbf{y}) - \frac{1}{4\pi}\log\frac{1}{c} + \frac{1}{c}\eta(c\mathbf{x})\right) v_c(\mathbf{x})v_c(\mathbf{y}). \tag{2.5.70}$$

We determine the limiting value of (2.5.70) as $c \to \infty$. Since $\int_{\Omega_c} v_c = 1$

$$\begin{aligned}
&\left|\int_{\Omega_c}\int_{\Omega_c}\left(\frac{1}{2}h(c\mathbf{x}, c\mathbf{y}) - \frac{1}{4\pi}\log\frac{1}{c}\right) v_c(\mathbf{x})v_c(\mathbf{y})d\mathbf{x}d\mathbf{y} - \frac{1}{2}h_0(\hat{\mathbf{X}}, \hat{\mathbf{X}})\right| \\
\leq &\sup_{\mathbf{x},\mathbf{y}\in B_{Ra/c}(\tilde{\mathbf{X}}_c)}\left|\frac{1}{2}h(c\mathbf{x}, c\mathbf{y}) - \frac{1}{4\pi}\log\frac{1}{c} - \frac{1}{2}h_0(\hat{\mathbf{X}}, \hat{\mathbf{X}})\right|.
\end{aligned}$$

Using the fact that $|\tilde{X}_{c2} - \hat{X}_2| \to 0$ as $c \to \infty$ and $c\tilde{X}_{c2} \to \infty$ as $c \to \infty$ we have

$$\begin{aligned}
&\sup_{\mathbf{x},\mathbf{y}\in B_{Ra/c}(\tilde{\mathbf{X}}_c)}\left|\frac{1}{2}h_1(c\mathbf{x}, c\mathbf{y}) - \frac{1}{4\pi}\log\frac{1}{c} - \frac{1}{2}h_0(\hat{\mathbf{X}}, \hat{\mathbf{X}})\right| \\
\leq &\sup_{\mathbf{x},\mathbf{y}\in B_{Ra/c}(\tilde{\mathbf{X}}_c)}\left|\frac{1}{2}\hat{h}_1(c\mathbf{x}, c\mathbf{y})\right| + \sup_{\mathbf{x},\mathbf{y}\in B_{Ra/c}(\tilde{\mathbf{X}}_c)}\left|\frac{1}{4\pi}\log\frac{2\hat{X}_2}{|\mathbf{x}-\bar{\mathbf{y}}|}\right| \\
&\to 0 \quad \text{as } c \to \infty \text{ by (2.3.10)}
\end{aligned}$$

and

$$\begin{aligned}
\sup_{\mathbf{x},\mathbf{y}\in B_{Ra/c}(\tilde{\mathbf{X}}_c)}\left|\frac{1}{2}h_0(c\mathbf{x}, c\mathbf{y}) - \frac{1}{4\pi}\log\frac{1}{c} - \frac{1}{2}h_0(\hat{\mathbf{X}}, \hat{\mathbf{X}})\right| &= \sup_{\mathbf{x},\mathbf{y}\in B_{Ra/c}(\tilde{\mathbf{X}}_c)}\left|\frac{1}{4\pi}\log\frac{2\hat{X}_2}{|\mathbf{x}-\bar{\mathbf{y}}|}\right| \\
&\to 0 \quad \text{as } c \to \infty
\end{aligned}$$

hence by (2.3.6)

$$\left|\int_{\Omega_c}\int_{\Omega_c}\left(\frac{1}{2}h(c\mathbf{x}, c\mathbf{y}) - \frac{1}{4\pi}\log\frac{1}{c}\right) v_c(\mathbf{x})v_c(\mathbf{y})d\mathbf{x}d\mathbf{y} - \frac{1}{2}h_0(\hat{\mathbf{X}}, \hat{\mathbf{X}})\right| \to 0 \quad \text{as } c \to \infty. \tag{2.5.71}$$

Also

$$\begin{aligned}
\left|\int_{\Omega_c}\frac{1}{c}\eta(c\mathbf{x})v_c(\mathbf{x})d\mathbf{x} - \hat{X}_2\right| &\leq \sup_{\mathbf{x}\in B_{Ra/c}(\tilde{\mathbf{X}}_c)}\left|\frac{1}{c}\eta(c\mathbf{x}) - \hat{X}_2\right| \\
&\leq \sup_{\mathbf{x}\in B_{Ra/c}(\tilde{\mathbf{X}}_c)}\frac{x_2}{c^2|\mathbf{x}|^2} + \frac{Ra}{c} + |\tilde{X}_{c2} - \hat{X}_2| \\
&\leq \frac{1}{c(c\tilde{X}_{c2} - Ra)} + \frac{Ra}{c} + |\tilde{X}_{c2} - \hat{X}_2| \\
&\to 0 \quad \text{as } c \to \infty.
\end{aligned} \tag{2.5.72}$$

Combining (2.5.70), (2.5.71) and (2.5.72)

$$\tilde{\Psi}_\lambda(v_c) = \frac{1}{4\pi}\log c + \mathcal{I}(v) - H_0(\hat{\mathbf{X}}) + o(1) \quad \text{as } c \to \infty. \tag{2.5.73}$$

In particular, since

$$\tilde{\zeta}_c(\mathbf{x}) = \begin{cases} c^2 \tilde{f}_\lambda(c(\mathbf{x} - \tilde{\mathbf{X}}_c)) & \mathbf{x} \in B_{Ra/c}(\tilde{\mathbf{X}}_c) \\ 0 & \mathbf{x} \in \Omega_c(\xi_0) \backslash B_{Ra/c}(\tilde{\mathbf{X}}_c) \end{cases}$$

we have

$$\tilde{\Psi}_\lambda(\tilde{\zeta}_c) = \frac{1}{4\pi}\log c + \mathcal{I}(\tilde{f}_\lambda) - H_0(\hat{\mathbf{X}}) + o(1) \quad \text{as } c \to \infty. \tag{2.5.74}$$

Let $w \in \mathcal{S}(\omega_0)$. Then $\tilde{\Psi}_\lambda(\tilde{\zeta}_c) \geq \tilde{\Psi}_\lambda(w_c)$ since w_c is a rearrangement of $\tilde{\zeta}_c$, hence by (2.5.73) and (2.5.74)

$$\mathcal{I}(\tilde{f}_\lambda) \geq \mathcal{I}(w) + o(1) \qquad \text{as } c \to \infty \tag{2.5.75}$$

for all $w \in \mathcal{S}(\omega_0)$.

We now show that $\mathcal{I}$ is strictly convex over $\mathcal{A}$.

Let $0 < \alpha < 1$ and let $v, u \in \mathcal{A}$ with $v \neq u$. Then by (2.5.73) and strict convexity of $\tilde{\Psi}_\lambda$

$$\begin{aligned} \mathcal{I}(\alpha v + (1-\alpha)u) &= \tilde{\Psi}_\lambda(\alpha v_c + (1-\alpha)u_c) - \frac{1}{4\pi}\log c + H_0(\hat{\mathbf{X}}) + o(1) \\ &< \alpha\tilde{\Psi}_\lambda(v_c) + (1-\alpha)\tilde{\Psi}_\lambda(u_c) - \frac{1}{4\pi}\log c + H_0(\hat{\mathbf{X}}) + o(1) \\ &= \alpha\mathcal{I}(v) + (1-\alpha)\mathcal{I}(u) + o(1) \qquad \text{as } c \to \infty. \end{aligned}$$

Letting $c \to \infty$ shows that $\mathcal{I}$ is strictly convex on $\mathcal{A}$.

Suppose, seeking a contradiction, that

$$\tilde{f}_\lambda \not\to \hat{\omega} \quad \text{in } L^p(B_{Ra}(0)) \text{ as } \lambda \to 0. \tag{2.5.76}$$

Then there exists $\varepsilon > 0$ and a subsequence $\tilde{f}_\lambda$ with

$$||\tilde{f}_\lambda - \hat{\omega}||_p > \varepsilon \tag{2.5.77}$$

and by weak sequential compactness we have, extracting a further subsequence if necessary, $\tilde{f}_\lambda \xrightarrow{w} f \in \overline{\mathcal{S}(\omega_0)}^w \subset \mathcal{A}$. Then $\mathcal{I}(\tilde{f}_\lambda) \to \mathcal{I}(f)$ as $\lambda \to 0$ by weak sequential continuity of $\mathcal{I}$ and by (2.5.75) $\mathcal{I}(f) \geq \mathcal{I}(w)$ for all $w \in \mathcal{S}(\omega_0)$.

We claim that $f \in \mathcal{S}(\omega_0)$. Suppose $f \notin \mathcal{S}(\omega_0)$. Since $\mathcal{I}$ is strictly convex over $\mathcal{A}$ (which is convex) and Gateaux differentiable on $L^p(B_{Ra}(0))$, [17, Proposition 5.4] yields

$$\mathcal{I}(v) - \mathcal{I}(w) > \langle \mathcal{K}w, v - w\rangle \quad \forall v, w \in \mathcal{A} \text{ with } v \neq w. \tag{2.5.78}$$

Let $u \in \mathcal{S}(\omega_0)$ maximise $\mathcal{K}f$ relative to $\overline{\mathcal{S}(\omega_0)}^w$ (by [9, Theorems 1 and 4] any bounded linear functional on $L^p(B_{Ra}(0))$ attains a maximum relative to $\mathcal{S}(\omega_0)$ hence such a u may be chosen). Then since $u \neq f$

$$\mathcal{I}(u) - \mathcal{I}(f) > \langle \mathcal{K}f, u - f\rangle \geq 0$$

which is a contradiction. Hence $f \in \mathcal{S}(\omega_0)$.

By [12, Lemma 2.6] the relative weak and strong topologies on $\mathcal{S}(\omega_0)$ coincide, hence $\tilde{f}_\lambda \to f$ strongly in $L^p(B_{Ra}(0))$. Therefore

$$\int_{B_{Ra}(0)}\int_{B_{Ra}(0)} \log\frac{1}{|\mathbf{x}-\mathbf{y}|} f(\mathbf{x})f(\mathbf{y})d\mathbf{x}d\mathbf{y} \geq \int_{B_{Ra}(0)}\int_{B_{Ra}(0)} \log\frac{1}{|\mathbf{x}-\mathbf{y}|} w(\mathbf{x})w(\mathbf{y})d\mathbf{x}d\mathbf{y}$$

for all $w \in \mathcal{S}(\omega_0)$ where $f \in \mathcal{S}(\omega_0)$. By [28, Lemma 3], $f(\mathbf{x}) = \hat{\zeta}(\mathbf{x} - \mathbf{z})$ for some $\mathbf{z} \in B_{Ra}(0)$. Since $\int_{B_{Ra}(0)} \mathbf{x}\tilde{f}_\lambda(\mathbf{x})d\mathbf{x} = 0$ for all λ we conclude $f = \hat{\omega}$. This contradicts (2.5.77). □

For $\mathbf{x} \in \mathbb{R}^2$ define

$$V(\mathbf{x}) = \mathcal{K}\hat{\omega}(\mathbf{x}) = \frac{1}{2\pi}\int_{B_a(0)} \log\frac{1}{|\mathbf{x}-\mathbf{y}|}\hat{\omega}(\mathbf{y})d\mathbf{y}.$$

LEMMA 2.5.11 *V is circularly symmetric decreasing (strictly) relative to the origin.*

Proof By rotational symmetry it is sufficient to show that

$$0 \leq x_1 < x_1' \Rightarrow V((x_1, 0)) > V((x_1', 0)). \tag{2.5.79}$$

For $x_1 \geq 0$ we define

$$W(x_1) = V((x_1, 0)) = \frac{1}{2\pi} \int_{B_a(0)} \log \frac{1}{|(x_1, 0) - \mathbf{y}|} \hat{\omega}(\mathbf{y}) d\mathbf{y}.$$

Then

$$W_{x_1}(x_1) = -\frac{1}{2\pi} \int_{B_a(0)} \frac{x_1 - y_1}{(x_1 - y_1)^2 + y_2^2} \hat{\omega}(\mathbf{y}) d\mathbf{y}.$$

We show that $W_{x_1}(x_1) < 0$. If $x_1 \geq a$ then obviously $W_{x_1}(x_1) < 0$. If $0 < x_1 < a$ then following the method of Turkington [36, Lemma 4.1] we define

$$A_{x_1} = \{(y_1, y_2) \in B_a(0) | y_1 > x_1\}$$

and

$$A'_{x_1} = \{(y_1, y_2) \in \mathbb{R}^2 | (2x_1 - y_1, y_2) \in A_{x_1}\} \subset B_a(0).$$

For $\mathbf{y} \in A_{x_1}$ we denote the reflected point as $\mathbf{y}' = (y_1', y_2') = (2x_1 - y_1, y_2)$. Then for all $\mathbf{y} \in A_{x_1}$ we have $\hat{\omega}(\mathbf{y}') \geq \hat{\omega}(\mathbf{y})$ since $\hat{\omega}$ is circularly symmetric decreasing, hence

$$\begin{aligned} & \int_{A_{x_1} \cup A'_{x_1}} \frac{x_1 - y_1}{(x_1 - y_1)^2 + y_2^2} \hat{\omega}(\mathbf{y}) d\mathbf{y} \\ = & \int_{A_{x_1}} \frac{x_1 - y_1}{(x_1 - y_1)^2 + y_2^2} (\hat{\omega}(\mathbf{y}) - \hat{\omega}(\mathbf{y}')) d\mathbf{y} > 0 \end{aligned}$$

and $W_{x_1}(x_1) < 0$. This completes the proof. □

Define

$$v_\lambda(\mathbf{x}) = (T\tilde{\omega}_\lambda - \lambda\eta - \gamma_\lambda)(c\tilde{\mathbf{X}}_c + \mathbf{x}) \quad \text{for } \mathbf{x} \in \mathbb{R}^2 \text{ with } c\tilde{\mathbf{X}}_c + \mathbf{x} \in \Omega.$$

THEOREM 2.5.12 *Let $R_1 > R$ and let $b = V((0, a))$. Then*

$$v_\lambda \to V - b \quad in\ C^1(\overline{B_{R_1 a}(0)})$$

as $\lambda \to 0$.

Proof Let $B = B_{R_1 a}(0)$. Define

$$V_\lambda(\mathbf{x}) = \mathcal{K}\tilde{f}_\lambda(\mathbf{x}) = \frac{1}{2\pi} \int_{B_{Ra}(0)} \log \frac{1}{|\mathbf{x} - \mathbf{y}|} \tilde{f}_\lambda(\mathbf{y}) d\mathbf{y}$$

to be the Newtonian potential of $\tilde{f}_\lambda$ where $\tilde{f}_\lambda$ is as defined in (2.5.69). From the result in Appendix B and the embedding $W^{2,p}(B) \to C^1(\overline{B})$ it follows that $\mathcal{K} : L^p(B) \to C^1(\overline{B})$ is bounded and Theorem 2.5.10 yields

$$V_\lambda \to V \quad \text{in } C^1(\overline{B}) \text{ as } \lambda \to 0. \tag{2.5.80}$$

Note that

$$v_\lambda(\mathbf{x}) = (T\tilde{\omega}_\lambda - \lambda\eta - \gamma_\lambda)(c\tilde{\mathbf{X}}_c + \mathbf{x}) = (T_c\tilde{\zeta}_c - \eta_c - \gamma_c)(\tilde{\mathbf{X}}_c + \frac{\mathbf{x}}{c})$$

and

$$\begin{aligned}
& V_\lambda(\mathbf{x}) - v_\lambda(\mathbf{x}) \\
= \; & \frac{1}{2\pi}\int_{B_{Ra}(0)} \log\frac{1}{|\mathbf{x}-\mathbf{y}|}\tilde{\omega}_\lambda(c\tilde{\mathbf{X}}_c + \mathbf{y})dy - \frac{1}{2\pi}\int_{B_{Ra}(c\tilde{\mathbf{X}}_c)} \log\frac{1}{|c\tilde{\mathbf{X}}_c + \mathbf{x} - \mathbf{y}|}\tilde{\omega}_\lambda(\mathbf{y})dy \\
& + \int_{B_{Ra}(c\tilde{\mathbf{X}}_c)} h(c\tilde{\mathbf{X}}_c + \mathbf{x}, \mathbf{y})\tilde{\omega}_\lambda(\mathbf{y})dy + \eta_c(\tilde{\mathbf{X}}_c + \frac{\mathbf{x}}{c}) + \gamma_c \\
= \; & \int_{B_{Ra}(c\tilde{\mathbf{X}}_c)} h(c\tilde{\mathbf{X}}_c + \mathbf{x}, \mathbf{y})\tilde{\omega}_\lambda(\mathbf{y})dy + \eta_c(\tilde{\mathbf{X}}_c + \frac{\mathbf{x}}{c}) + \gamma_c.
\end{aligned}$$

Hence

$$\begin{aligned}
|V_\lambda(\mathbf{x}) - v_\lambda(\mathbf{x})| & \le \sup_{\mathbf{y}\in B_{Ra}(c\tilde{\mathbf{X}}_c)} \left|h(c\tilde{\mathbf{X}}_c + \mathbf{x}, \mathbf{y}) + \gamma_c\right| + |\eta_c(\tilde{\mathbf{X}}_c + \frac{\mathbf{x}}{c})| \\
& \le \sup_{\mathbf{y}\in B_{Ra}(c\tilde{\mathbf{X}}_c)} |\hat{h}(c\tilde{\mathbf{X}}_c + \mathbf{x}, \mathbf{y})| + \left|h_0(c\tilde{\mathbf{X}}_c + \mathbf{x}, \mathbf{y}) + \gamma_c\right| \\
& \quad + |\eta_c(\tilde{\mathbf{X}}_c + \frac{\mathbf{x}}{c})| \qquad (2.5.81) \\
& \le \sup_{\mathbf{y}\in B_{Ra}(c\tilde{\mathbf{X}}_c)} |\hat{h}_1(c\tilde{\mathbf{X}}_c + \mathbf{x}, \mathbf{y})| + \left|h_0(c\tilde{\mathbf{X}}_c + \mathbf{x}, \mathbf{y}) + \gamma_c\right| \\
& \quad + |\eta_c(\tilde{\mathbf{X}}_c + \frac{\mathbf{x}}{c})|. \qquad (2.5.82)
\end{aligned}$$

Let

$$G_c = \{\mathbf{x} \in \mathbb{R}^2 | \tilde{\mathbf{X}}_c + \frac{\mathbf{x}}{c} \in \Omega_c(\xi_0)\}$$

and choose c sufficiently large that

$$\frac{1}{8\pi} \le dist(B_{R_1 a/c}(\tilde{\mathbf{X}}_c), \partial\Omega_c(\xi_0)). \tag{2.5.83}$$

Then for all $\mathbf{x} \in \partial G_c$ we have $c/8\pi \le |\mathbf{x}| \le 2c\xi_0$ and $|c\tilde{\mathbf{X}}_c + \mathbf{x} - \bar{\mathbf{y}}| \le 2c\xi_0$ for all $\mathbf{y} \in B_{Ra}(c\tilde{\mathbf{X}}_c)$. Therefore by (2.5.42)

$$h_0(c\tilde{\mathbf{X}}_c+\mathbf{x},\mathbf{y})+\gamma_c \ge \frac{1}{2\pi}\log\frac{1}{2c\xi_0}+\frac{1}{2\pi}\log\frac{c}{2a}+O(1) = \frac{1}{2\pi}\log\frac{1}{4a\xi_0}+O(1) \quad \text{as } c\to\infty \tag{2.5.84}$$

Also

$$\begin{aligned}
\gamma_c &\le 2\tilde{\Psi}_\lambda(\tilde{\zeta}_c) + \int_{\Omega_c} \eta_c\tilde{\zeta}_c \\
&\le \int_{\Omega_c} \tilde{\zeta}_c T_c \tilde{\zeta}_c \\
&= \int_{\Omega} \tilde{\omega}_\lambda T \tilde{\omega}_\lambda \\
&\le \int_{\Omega} \tilde{\omega}_\lambda T_0 \tilde{\omega}_\lambda \\
&\le \int_{\Omega} (M_1 + M_2|\log x_2|)\tilde{\omega}_\lambda \quad \text{by Lemma 2.3.1 (i)} \\
&\le M_3\log(3c/8\pi)
\end{aligned}$$

if c is sufficiently large where M_1, M_2 and M_3 are positive constants independent of c.

If $\mathbf{x} \in \partial G_c$ and $\mathbf{y} \in B_{Ra}(c\tilde{\mathbf{X}}_c)$ then $|c\tilde{\mathbf{X}}_c + \mathbf{x} - \bar{\mathbf{y}}| \ge c/8\pi$ and

$$h_0(c\tilde{\mathbf{X}}_c + \mathbf{x}, \mathbf{y}) + \gamma_c \le \frac{1}{2\pi}\log\frac{8\pi}{c} + M_3\log\frac{3c}{8\pi} \le M_3\log c \tag{2.5.85}$$

for c sufficiently large.

If $\mathbf{x} \in \partial G_c$ with $c\tilde{\mathbf{X}}_c + \mathbf{x} \in \partial\Omega$ then $\hat{h}(c\tilde{\mathbf{X}}_c + \mathbf{x}, \mathbf{y}) = 0$. On the other hand for $\mathbf{x} \in \partial G_c$ with $c\tilde{\mathbf{X}}_c + \mathbf{x} \notin \partial\Omega$ we have $|c\tilde{\mathbf{X}}_c + \mathbf{x}| = c\xi_0$ and

$$\sup_{\mathbf{y}\in B_{Ra}(c\tilde{\mathbf{X}}_c)} |\hat{h}_1(c\tilde{\mathbf{X}}_c + \mathbf{x}, \mathbf{y})| \le \sup_{\mathbf{y}\in B_{Ra}(c\tilde{\mathbf{X}}_c)} \frac{(c\tilde{X}_{c2} + x_2)y_2}{\pi(|c\tilde{\mathbf{X}}_c + \mathbf{x}||\mathbf{y}| - 1)^2} \to 0 \qquad \text{as } c\to\infty. \tag{2.5.86}$$

Finally

$$0 \le \eta_c(\tilde{\mathbf{X}}_c + \frac{\mathbf{x}}{c}) \le \tilde{X}_{c2} + \frac{x_2}{c} \le \xi_0 \tag{2.5.87}$$

for all $\mathbf{x} \in \partial G_c$. Combining the estimates (2.5.84), (2.5.85), (2.5.86) and (2.5.87) with (2.5.81) and (2.5.82) we have

$$\begin{aligned} -\Delta(V_\lambda - v_\lambda) &= 0 \text{ almost everywhere in } G_c \\ |V_\lambda - v_\lambda| &\le M_4 \log c \text{ on } \partial G_c \end{aligned} \tag{2.5.88}$$

for c sufficiently large where M_4 is a positive constant.

Applying an interior gradient estimate [23, Section 2.7]

$$\sup_{\mathbf{x}\in\overline{B}} |\nabla V_\lambda(\mathbf{x}) - \nabla v_\lambda(\mathbf{x})| \le \frac{\text{const.}}{dist(B, \partial G_c)} \sup_{\mathbf{x}\in G_c} |V_\lambda(\mathbf{x}) - v_\lambda(\mathbf{x})| \le \frac{\text{const.}\log c}{c} \tag{2.5.89}$$

for c sufficiently large.

We now show that $V_\lambda - v_\lambda \to b$ in $C^1(\overline{B})$ as $\lambda \to 0$, where $b = V(0, a)$ is a constant.

Suppose not. Then there exists $\varepsilon > 0$ and a subsequence $(V_\lambda - v_\lambda)$ such that

$$|(V_\lambda - v_\lambda) - b|_{1;B} > \varepsilon. \tag{2.5.90}$$

Since $\{v_\lambda > 0\} = \{\tilde{f}_\lambda > 0\} \subset B_{Ra}(0)$ we deduce that for each $\lambda > 0$ there exists $y_\lambda \in B$ with $v_\lambda(y_\lambda) = 0$. Then for $\mathbf{x} \in \overline{B}$ by the Mean Value Theorem and (2.5.89)

$$|(V_\lambda - v_\lambda)(\mathbf{x}) - V_\lambda(y_\lambda)| = |(V_\lambda - v_\lambda)(\mathbf{x}) - (V_\lambda - v_\lambda)(y_\lambda)| \to 0 \quad \text{as } c \to \infty.$$

From (2.5.80) we deduce $(V_\lambda - v_\lambda)$ is bounded on $\overline{B}$ for sufficiently large c.

Choose a subsequence $(V_\lambda - v_\lambda)$ such that for some $\mathbf{z} \in B$, $(V_\lambda - v_\lambda)(\mathbf{z})$ converges to $\tilde{b}$, say. Then $(V_\lambda - v_\lambda)$ converges to $\tilde{b}$ in $C(\overline{B})$ and by (2.5.89) $(V_\lambda - v_\lambda) \to \tilde{b}$ in $C^1(\overline{B})$. From (2.5.80) we deduce $v_\lambda \to V - \tilde{b}$ in $C^1(\overline{B})$ as $c \to \infty$.

It remains to show $\tilde{b} = V(0, a)$. Suppose $\tilde{b} > V(0, a)$ and let $\delta = \tilde{b} - V(0, a) > 0$. Then $|v_\lambda - (V - \tilde{b})| < \delta/2$ for c sufficiently large. By Lemma 2.5.11, V is strictly symmetric decreasing hence

$$\{v_\lambda > 0\} \subset \{V - \tilde{b} > -\delta/2\} \subset B_r(0)$$

for some $r < a$ which contradicts the fact that $\mu_2\{v_\lambda > 0\} = \pi a^2$ for all $\lambda < \lambda_0$.

If $\tilde{b} < V(0,a)$ a similar argument yields a contradiction. Hence $\tilde{b} = V(0,a)$. This contradicts (2.5.90) and we conclude that the original sequence $V_\lambda - v_\lambda \to b$ in $C^1(\overline{B})$. □

COROLLARY 2.5.13 *Let $R_1 > R$ and $B = B_{R_1 a}(0)$. Let*

$$Y_\lambda = \{\mathbf{x} \in \overline{B} | v_\lambda(\mathbf{x}) \geq 0\}$$

and

$$Z = \{\mathbf{x} \in \overline{B} | V(\mathbf{x}) \geq b\} = \overline{B_a(0)}.$$

Then

$$d(Y_\lambda, Z) \to 0 \quad \text{as } \lambda \to 0$$

where d is the Hausdorff distance on the nonempty closed bounded subsets of $\overline{B}$.

Proof The Hausdorff distance between Y_λ and Z is defined as

$$d(Y_\lambda, Z) = \max\{\sup_{\mathbf{y} \in Y_\lambda} dist(y, Z), \sup_{\mathbf{z} \in Z} dist(z, Y_\lambda)\}.$$

Let $\delta > 0$. Then by Lemma 2.5.11 there exists $\varepsilon_1 > 0$ such that

$$|\mathbf{x}| > a + \delta \Rightarrow V(\mathbf{x}) - b < -\varepsilon_1$$

and by Theorem 2.5.12 there exists C_1 such that for $c > C_1$

$$|v_\lambda(\mathbf{x}) - (V(\mathbf{x}) - b)| < \varepsilon_1/2 \;\; \forall \mathbf{x} \in B.$$

Hence $Y_\lambda \subset B_{a+\delta}(0)$ for $c > C_1$.

Similarly there exists C_2 such that $B_{a-\delta}(0) \subset Y_\lambda$ for $c > C_2$. The result is now immediate. □

We conclude with some asymptotic estimates.

COROLLARY 2.5.14 *Let $\tilde{\omega}_\lambda$ be a sequence of maximisers of Ψ_λ relative to $\mathcal{F}$. Then*

$$\Psi_\lambda(\tilde{\omega}_\lambda) \quad = \quad \frac{1}{4\pi}\log c + \mathcal{I}(\hat{\omega}) - H_0(\hat{\mathbf{X}}) + o(1) \qquad \text{as } c \to \infty, \qquad (2.5.91)$$

$$\begin{aligned} F(\tilde{\omega}_\lambda) &:= \frac{1}{2}\int_{B_{Ra}(0)} |\nabla v_\lambda^+|^2 \\ &= \mathcal{I}(\hat{\omega}) - \frac{1}{2}V(0,a) + o(1) \qquad \textit{as } c \to \infty \end{aligned} \tag{2.5.92}$$

and

$$\gamma_\lambda = \frac{1}{2\pi}\log c - 2H_0(\hat{\mathbf{X}}) + V(0,a) + \frac{1}{4\pi} + o(1) \qquad \textit{as } c \to \infty. \tag{2.5.93}$$

Proof The estimate (2.5.91) follows immediately from (2.5.74) and Theorem 2.5.10 on observing $\Psi_\lambda(\tilde{\omega}_\lambda) = \tilde{\Psi}_\lambda(\tilde{\zeta}_c)$.

For λ sufficiently small $v_\lambda^+ \in W_0^{1,p}(B_{Ra}(0))$ and

$$\nabla v_\lambda^+ = \begin{cases} \nabla v_\lambda & \text{almost everywhere on } v_\lambda > 0, \\ 0 & \text{almost everywhere on } v_\lambda \leq 0. \end{cases}$$

Therefore by Theorem 2.5.12

$$\begin{aligned} \int_{B_{Ra}(0)} |\nabla v_\lambda^+|^2 - |\nabla (V-b)^+|^2 &= \int_{\{v_\lambda>0\}\cap\{V-b>0\}} |\nabla v_\lambda|^2 - |\nabla(V-b)|^2 \\ &\quad + \int_{\{v_\lambda>0\}\cap\{V-b\leq 0\}} |\nabla v_\lambda|^2 + \int_{\{v_\lambda\leq 0\}\cap\{V-b>0\}} |\nabla(V-b)^+|^2 \\ &\quad \to 0 \quad \text{as } \lambda \to 0. \end{aligned}$$

Hence

$$F(\tilde{\omega}_\lambda) = \frac{1}{2}\int_{B_{Ra}(0)} |\nabla v_\lambda^+|^2 \to \frac{1}{2}\int_{B_{Ra}(0)} |\nabla(V-b)^+|^2 = \frac{1}{2}\int_{B_a(0)} |\nabla V|^2 \tag{2.5.94}$$

as $\lambda \to 0$.

Note that

$$F(\tilde{\omega}_\lambda) = \frac{1}{2}\int_\Omega (T\tilde{\omega}_\lambda - \lambda\eta - \gamma_\lambda)\tilde{\omega}_\lambda$$

hence

$$\begin{aligned} \gamma_\lambda &= 2\Psi_\lambda(\tilde{\omega}_\lambda) - 2F(\tilde{\omega}_\lambda) + \lambda\int_\Omega \eta\tilde{\omega}_\lambda \\ &= \frac{1}{2\pi}\log c + 2\mathcal{I}(\hat{\omega}) - 2H_0(\hat{\mathbf{X}}) - \int_{B_a(0)} |\nabla V|^2 + \frac{1}{4\pi} + o(1) \end{aligned} \tag{2.5.95}$$

as $c \to \infty$. Also, by the Divergence Theorem,

$$2\mathcal{I}(\hat{\omega}) - \mathcal{K}\hat{\omega}(0,a) = \int_{B_a(0)} \hat{\omega}(\mathcal{K}\hat{\omega} - \mathcal{K}\hat{\omega}(0,a)) = \int_{B_a(0)} |\nabla K\hat{\omega}|^2 = \int_{B_a(0)} |\nabla V|^2.$$

Combining this with (2.5.94) and (2.5.95) we obtain (2.5.92) and (2.5.93). □

2.5.4 Physical Interpretation

Let $\psi = \psi_\lambda - \lambda\eta$ where ψ_λ is as in Theorem 2.5.6. Since $\psi = 0$ on $\partial\Omega$ we can extend ψ as an odd function of x_2 to a function ψ^* on Ω^*.

By Lemma 2.5.7 the support of $\tilde{\omega}_\lambda$ is bounded away from the x_1-axis if λ is sufficiently small hence ψ^* is harmonic in a neighbourhood of the x_1-axis. By Theorem 2.5.6 $-\Delta\psi^*$ is locally a function of ψ^* in $x_2 > 0$ and by symmetry this is also true in $x_2 < 0$. Hence $-\Delta\psi^*$ is locally a function of ψ^* in Ω^* and therefore the stream function for a steady flow of an ideal fluid. The velocity is given by

$$\left(\frac{\partial\psi^*}{\partial x_2}, -\frac{\partial\psi^*}{\partial x_1}\right).$$

Since $|\nabla\psi_\lambda(\mathbf{x})| \to 0$ as $|\mathbf{x}| \to \infty$ the flow tends to a uniform stream with velocity $(-\lambda, 0)$ at infinity.

We include a brief summary of the results in [11, Section 5]. The Euler equations for fluid flow are

$$(\mathbf{u}.\nabla)\mathbf{u} = -\nabla P \tag{2.5.96}$$

$$\nabla.\mathbf{u} = 0 \tag{2.5.97}$$

where $\mathbf{u}$ is the velocity field and P is the pressure.

The stream function $\psi \in W^{2,p}_{loc}(\Omega^*)$ satisfies

$$\begin{aligned} -\Delta\psi &= \phi^+(\psi) \text{ almost everywhere in } x_2 > 0, \\ -\Delta\psi &= \phi^-(\psi) \text{ almost everywhere in } x_2 < 0 \end{aligned}$$

where ϕ^+ is an increasing function and $\phi^-(s) = -\phi^+(-s)$ for all $-s \in \operatorname{dom} \phi^+$.

Since the second order cross derivatives of ψ are equal almost everywhere equation (2.5.97) holds almost everywhere. The pressure P is defined by

$$-P(\mathbf{x}) = \begin{cases} \frac{1}{2}|\nabla\psi|^2 + \Phi^+(\psi) \text{ for } x_2 \geq 0 \\ \frac{1}{2}|\nabla\psi|^2 + \Phi^-(\psi) \text{ for } x_2 \leq 0 \end{cases}$$

where Φ^+ and Φ^- are the respective indefinite integrals of ϕ^+ and ϕ^- satisfying $\Phi^+(0) = 0 = \Phi^-(0)$ and (2.5.96) is satisfied almost everywhere.

Chapter 3

Vortex rings

3.1 Introduction

We prove the existence of steady vortex rings in an ideal fluid occupying the whole of $\mathbb{R}^3$. The variational principle used was proposed by Benjamin [6] and involves the maximisation of a functional relative to the weak closure of the set of rearrangements of a prescribed function.

The vortex core (the region where the vorticity, ω, is positive) is a cylindrically symmetric bounded set. The quantity ω/r is a rearrangement of a curtailment of a prescribed non-negative function $\zeta_0 \in L^p(\Pi)$ ($p > 5/2$) having bounded support. The flow approaches a uniform stream at infinity.

In view of the cylindrical symmetry we work in a half-plane, Π, defined by

$$\Pi = \{(r, z) \in \mathbb{R}^2 | r > 0\}$$

and endow Π with the measure ν having density $2\pi r$ with respect to plane Lebesgue measure. Define a differential operator $\mathcal{L}$ in Π by

$$\mathcal{L}u = -\frac{1}{r}\left(\frac{1}{r}u_r\right)_r - \frac{1}{r^2}u_{zz}.$$

We prove the following

THEOREM 3.1.1 *Let $\zeta_0 \in L^p(\Pi)$, $5 < p < \infty$, be non-zero, non-negative and vanish outside a set of finite ν-measure. Then for each $\lambda > 0$, there is a function u_λ satisfying* *(i)* $u_\lambda \in W^{2,p}_{loc}(\Pi)$ *and*

$$\mathcal{L}u_\lambda = \phi \circ (u_\lambda - \frac{1}{2}\lambda r^2)$$

almost everywhere in Π *for some increasing function* ϕ,

(ii) $\zeta_\lambda = \mathcal{L}u_\lambda$ *is a rearrangement of a curtailment of* ζ_0 *having bounded support and* ζ_λ *is a maximiser of a variational functional that will be defined in (3.2.2),*

(iii) u_λ *and* ζ_λ *are symmetric decreasing in* z,

(iv)

$$\zeta_\lambda^{-1}(0,\infty) \subset (u_\lambda - \frac{1}{2}\lambda r^2)^{-1}(0,\infty)$$

except for a set of measure zero and moreover if $\zeta_\lambda \notin R(\zeta_0)$ *then*

$$\zeta_\lambda^{-1}(0,\infty) = (u_\lambda - \frac{1}{2}\lambda r^2)^{-1}(0,\infty)$$

except for a set of measure zero.

The stream function for the flow is $u_\lambda - \lambda r^2/2$ with corresponding velocity

$$\mathbf{v} = (-r^{-1}u_z, 0, r^{-1}u_r - \lambda)$$

in cylindrical coordinates r, θ, z. At infinity the flow approaches a uniform flow with velocity λ in the negative z direction. Since

$$\text{curl } \mathbf{v} = (0, \omega, 0)$$

where ω is the vorticity, we have $\omega = r\mathcal{L}u_\lambda$ and ω/r is a rearrangement of a curtailment of ζ_0.

We believe that if λ is sufficiently small ζ_λ in the above theorem is actually a rearrangement of ζ_0 but we have been unable to prove this for arbitrary ζ_0. However, in the special case where ζ_0 is constant in its support we use the results of Amick and Fraenkel [3] to prove the existence of $\lambda_0 > 0$ such that for $0 < \lambda < \lambda_0$, ζ_λ is a rearrangement of ζ_0 and for $\lambda > \lambda_0$, ζ_λ is zero almost everywhere. When $\lambda = \lambda_0$, ζ_λ may be either a rearrangement of ζ_0 or zero.

3.2 Description of the Method

For $\xi > 0$ let

$$\Pi(\xi) = \{(r, z) \in \Pi | r < \xi\}.$$

Let $\zeta_0 \in L^p(\Pi)$ $(p > 5/2)$ be a non-negative, non-zero function vanishing outside a set of finite measure. A non-negative measurable function ζ defined on Π is a rearrangement (with respect to ν-measure) of ζ_0 if

$$\nu(\zeta^{-1}[\alpha, \infty)) = \nu(\zeta_0^{-1}[\alpha, \infty))$$

for all $\alpha \in \mathbb{R}$. Let $R(\zeta_0)$ denote the set of rearrangements of ζ_0 and let $\overline{R(\zeta_0)}^w$ denote the closure of $R(\zeta_0)$ in the L^q-topology where q denotes the conjugate exponent of p. $\overline{R(\zeta_0)}_b^w$ will denote those functions in $\overline{R(\zeta_0)}^w$ that have bounded support.

A non-negative measurable function u defined on Π will be called Steiner-symmetric if

$$0 \leq z \leq z' \Rightarrow u(r, -z) = u(r, z) \geq u(r, z') \geq 0.$$

There is an essentially unique rearrangement u^* of u called the Steiner-symmetrisation of u such that u^* is Steiner-symmetric and

$$\mu_1\{z | u(r, z) \geq \alpha\} = \mu_1\{z | u^*(r, z) \geq \alpha\}$$

for almost every r and every $\alpha > 0$.

For $v \in L^p(\Pi)$ define

$$Tv(r, z) = \int_\Pi g(r, z, r', z') v(r', z') \; 2\pi r' dr' dz'. \tag{3.2.1}$$

where $g(r, z, r', z')$ is the Green's function (see 3.3.17) for the operator $\mathcal{L}$ on Π (with ν measure).

For $\lambda > 0$ we consider the variational functional

$$\Psi_\lambda(v) = \frac{1}{2} \int_\Pi vTv \; d\nu - \frac{1}{2}\lambda \int_\Pi r^2 v \; d\nu. \tag{3.2.2}$$

We show that there exist $\xi(\lambda) > 0$ and a maximising sequence for Ψ_λ relative to $\overline{R(\zeta_0)}_b^w$ such that each function is Steiner-symmetric and supported in $\Pi(\xi(\lambda))$.

We then apply a result of Lions [29] regarding compact embeddings of function spaces with Steiner-symmetric elements to show that this maximum is attained. If ζ_λ is a maximiser then $u_\lambda = T\zeta_\lambda$ satisfies the properties stated in Theorem 3.1.1.

3.3 Preliminaries

3.3.1 Spaces appropriate to the study of $\mathcal{L}$ and T

In choosing a space appropriate to the study of the operator T we follow Amick and Fraenkel [3, Section 2.2].

Define H to be the completion of test functions on Π with the scalar product

$$\langle u, v\rangle_H = \int_\Pi \frac{1}{r^2}\nabla u.\nabla v \; d\nu.$$

We can regard Π as being the intersection of $\mathbb{R}^5$ with a half-plane bounded by the z-axis and we shall use r to denote the distance from the z-axis. Then functions defined almost everywhere on Π can be identified with cylindrically symmetric functions defined almost everywhere in $\mathbb{R}^5$. Formally we have

$$\mathcal{L}(r^2 u) = -\Delta_5 u$$

for functions on Π where Δ_5 is the 5-dimensional Laplacian.

Define E to be the completion of the test functions in $\mathbb{R}^5$ with the scalar product

$$\langle u, v\rangle_E = \int_{\mathbb{R}^5} \nabla u.\nabla v \; d\mu \tag{3.3.3}$$

where μ has density $1/\pi$ with respect to 5-dimensional Lebesgue measure. By [3, Lemma 2.1(a)] the space E is embedded in $L^{10/3}(\mathbb{R}^5)$ and

$$||u||_{10/3} \leq k_1 ||u||_E \tag{3.3.4}$$

where $k_1 = \frac{4}{3}(\frac{\pi}{5})^{1/2}$.

Let E_c denote the completion of the cylindrically symmetric test functions on $\mathbb{R}^5$ with the same scalar product as (3.3.3). Then by [3, Lemma 2.2] H and E_c are isometrically isomorphic under the transformation $\phi = r^2 u$ of any $\phi \in H$ or

$u \in E_c$.

Let τ denote the measure on Π with density $2\pi r^3$ with respect to Lebesgue measure.

For a measure σ on Π and an open subset Ω of Π we denote by $L^p(\Omega, \sigma)$ the completion of $C_0^\infty(\Omega)$ in the norm

$$||u||_{p;\Omega;\sigma} = \left(\int_\Omega |u|^p d\sigma\right)^{1/p}.$$

When $\sigma = \nu$ the label σ will be omitted and when also $\Omega = \Pi$ the label Ω will be omitted.

3.3.2 Inversion of $\mathcal{L}$

We follow Burton [10, Section 3.1] in defining an inverse operator, $K : L^p(\Pi, \tau) \to H$ for $\mathcal{L}$ and an inverse operator, $\mathcal{K} : L^{10/7}(\mathbb{R}^5) \to E$, for Δ_5.

Let $v \in L^p(\Pi, \tau)$. Define Kv to be the unique minimiser over $u \in H$ of the functional

$$\Phi_H^v(u) = \frac{1}{2}||u||_H^2 - \int_\Pi uv \ d\nu.$$

For $w \in L^{10/7}(\mathbb{R}^5)$ define $\mathcal{K}w$ as the unique minimiser over $u \in E$ of the functional

$$\Phi_E^w(u) = \frac{1}{2}||u||_E^2 - \int_{\mathbb{R}^5} uw \ d\mu.$$

We shall prove that if w is cylindrically symmetric then $\mathcal{K}w \in E_c$ and, for suitable p, $Kv = r^2\mathcal{K}w$ where v is the function on Π identified with w. Also, $\mathcal{K}$ is an integral operator with Newtonian kernel from which we deduce the equivalence of K and T on $L^p(\Pi(\xi))$ ($\subset L^p(\Pi, \tau)$) for each $\xi > 0$.

LEMMA 3.3.1 *For $w \in L^{10/7}(\mathbb{R}^5)$ $\mathcal{K}w$ is well-defined and $\mathcal{K} : L^{10/7}(\mathbb{R}^5) \to E$ is a bounded linear operator with*

$$||\mathcal{K}|| \leq \frac{2k_1}{\pi}. \tag{3.3.5}$$

Proof Recalling (3.3.4) we have

$$\begin{aligned}
\Phi_E^w(u) &= \frac{1}{2}||u||_E^2 - \int_{\mathbb{R}^5} uw \ d\mu \\
&\geq \frac{1}{2}||u||_E^2 - \frac{1}{\pi}||w||_{10/7}||u||_{10/3} \\
&\geq \frac{1}{2}||u||_E^2 - \frac{k_1}{\pi}||w||_{10/7}||u||_E \\
&\to \infty \quad \text{as } ||u||_E \to \infty.
\end{aligned}$$

Hence Φ_E^w is a strictly convex, weakly lower semicontinuous, coercive functional on E and there exists a unique minimiser $\mathcal{K}w$, say. The Fréchet derivative of the functional at $\mathcal{K}w$ is identically zero. Thus

$$\int_{\mathbb{R}^5} \nabla \mathcal{K}w . \nabla h \ d\mu = \int_{\mathbb{R}^5} hw \ d\mu \quad \forall h \in E$$

and, in particular,

$$\int_{\mathbb{R}^5} \mathcal{K}w(-\Delta\phi) \ d\mu = \int_{\mathbb{R}^5} \phi w \ d\mu \quad \forall \phi \in C_0^\infty(\mathbb{R}^5).$$

Therefore $-\Delta \mathcal{K}w = w$ in the distributional sense.

Considering $u = 0$ we have

$$\frac{1}{2}||\mathcal{K}w||_E^2 \leq \int_{\mathbb{R}^5} w\mathcal{K}w \ d\mu \leq \frac{1}{\pi}||w||_{10/7}||\mathcal{K}w||_{10/3} \leq (\frac{k_1}{\pi})||w||_{10/7}||\mathcal{K}w||_E$$

hence $\mathcal{K}$ is bounded. The linearity of $\mathcal{K}$ is easily verified. □

In identifying functions on Π with functions on $\mathbb{R}^5$ we shall denote by $\mathbf{x} = (x_1, ..., x_4, z) \in \mathbb{R}^5$ (where $\sum_{i=1}^4 x_i^2 = r^2$) the elements in $\mathbb{R}^5$ corresponding to $(r, z) \in \Pi$.

Let G denote the subgroup of $GL(\mathbb{R}, 5)$ consisting of all orthogonal transformations which act as the identity on the z-axis. Then G is a closed subspace of the orthogonal group hence G is a compact subgroup of $GL(\mathbb{R}, 5)$. Let $d\theta$ denote the element of Haar measure.

Let $u \in L^{10/3}(\mathbb{R}^5)$ and $\mathbf{x} \in \mathbb{R}^5$. Define

$$\mathcal{A}u(\mathbf{x}) = \int_G u(\theta \mathbf{x}) \ d\theta. \tag{3.3.6}$$

LEMMA 3.3.2 *Let $\phi \in C_0^\infty(\mathbb{R}^5)$. Then*

$$||\mathcal{A}\phi||_E \leq ||\phi||_E.$$

Proof Let $\phi \in C_0^\infty(\mathbb{R}^5)$. Then

$$\nabla(\mathcal{A}\phi)(\mathbf{x}) = \int_G \theta^T \nabla\phi(\theta\mathbf{x})\ d\theta.$$

We first prove by induction that $\mathcal{A}\phi \in C_0^\infty(\mathbb{R}^5)$. Suppose

$$\partial^\alpha(\mathcal{A}\phi)(\mathbf{x}) = \int_G \sum_{i=1}^{5^n} f_i(\theta)\partial^{\alpha_i}\phi(\theta\mathbf{x})\ d\theta$$

where $\alpha, \alpha_1, ..., \alpha_{5^n}$ are multi-indices of degree n and $f_1, ..., f_{5^n}$ are continuous functions. Then

$$\begin{aligned}
& \partial^\alpha(\mathcal{A}\phi)(\mathbf{x}+\mathbf{h}) - \partial^\alpha(\mathcal{A}\phi)(\mathbf{x}) \\
= & \int_G \sum_{i=1}^{5^n} f_i(\theta)\left[\partial^{\alpha_i}\phi(\theta\mathbf{x}+\theta\mathbf{h}) - \partial^{\alpha_i}\phi(\theta\mathbf{x})\right]\ d\theta \\
= & \int_G \sum_{i=1}^{5^n} f_i(\theta)\left[\nabla(\partial^{\alpha_i}\phi)(\theta\mathbf{x}).\theta\mathbf{h} + \frac{1}{2}(\theta\mathbf{h})^T H(\partial^{\alpha_i}\phi(t_{i,\theta}))(\theta\mathbf{h})\right]\ d\theta
\end{aligned}$$

where $H(\partial^{\alpha_i}\phi)$ is the Hessian matrix and $t_{i,\theta}$ is in the line segment joining $\theta\mathbf{x}$ and $\theta\mathbf{x}+\theta\mathbf{h}$. Therefore

$$\nabla(\partial^\alpha(\mathcal{A}\phi))(\mathbf{x}) = \int_G \sum_{i=1}^{5^n} f_i(\theta)\theta^T\nabla(\partial^{\alpha_i}\phi)(\theta\mathbf{x})\ d\theta$$

and

$$\partial_j\partial^\alpha(\mathcal{A}\phi)(\mathbf{x}) = \int_G \sum_{i=1}^{5^n} f_i(\theta)\sum_{k=1}^{5}\theta_{kj}(\partial_k\partial^{\alpha_i}\phi)(\theta\mathbf{x})\ d\theta \quad j = 1, ..., 5$$

where $\theta = (\theta_{ij})_{i,j=1}^5$. Hence

$$\partial_j\partial^\alpha(\mathcal{A}\phi)(\mathbf{x}) = \int_G \sum_{i'=1}^{5^{n+1}} \tilde{f}_{i'}(\theta)\partial^{\beta_{i'}}\phi(\theta\mathbf{x})\ d\theta \tag{3.3.7}$$

where $\beta_1, ..., \beta_{5^{n+1}}$ are multi-indices of degree $n+1$ and $\tilde{f}_1, ..., \tilde{f}_{5^{n+1}}$ are continuous functions. By induction $\mathcal{A}\phi \in C_0^\infty(\mathbb{R}^5)$.

Also

$$\begin{aligned}
||\mathcal{A}\phi||_E^2 &= \int_{\mathbb{R}^5} |\nabla(\mathcal{A}\phi)|^2 \ d\mu \\
&= \int_{\mathbb{R}^5} \left| \int_G \theta^T \nabla\phi(\theta\mathbf{x}) \right|^2 \ d\mu \\
&= \int_{\mathbb{R}^5} \left(\int_{\theta\in G} \int_{\psi\in G} (\theta^T \nabla\phi(\theta\mathbf{x}))^T (\psi^T \nabla\phi(\psi\mathbf{x})) \ d\psi d\theta \right) \ d\mu \\
&\leq \int_{\mathbb{R}^5} \left(\int_{(\theta,\psi)\in(G\times G)} |\nabla\phi(\theta\mathbf{x})|^2 \ d\psi d\theta \right)^{1/2} \left(\int_{(\theta,\psi)\in(G\times G)} |\nabla\phi(\psi\mathbf{x})|^2 \ d\psi d\theta \right)^{1/2} \ d\mu \\
&= \int_{\mathbb{R}^5} \left(\int_G |\nabla\phi(\theta\mathbf{x})|^2 \ d\theta \right) \ d\mu \\
&= \int_G \left(\int_{\mathbb{R}^5} |\nabla\phi(\theta\mathbf{x})|^2 \ d\mu \right) \ d\theta \\
&= \int_{\mathbb{R}^5} |\nabla\phi(\mathbf{x})|^2 \ d\mu. \ \square
\end{aligned}$$

LEMMA 3.3.3 $\mathcal{A} : L^{10/3}(\mathbb{R}^5) \to L^{10/3}(\mathbb{R}^5)$ *is a bounded linear operator.*

Proof For all $u \in L^{10/3}(\mathbb{R}^5)$ we have by Jensen's inequality

$$\begin{aligned}
\int_{\mathbb{R}^5} |\mathcal{A}u|^{10/3} \ d\mu_5 &= \int_{\mathbb{R}^5} \left| \int_G u(\theta\mathbf{x}) \ d\theta \right|^{10/3} \ d\mu_5 \\
&\leq \int_{\mathbb{R}^5} \int_G |u(\theta\mathbf{x})|^{10/3} \ d\theta d\mu_5 \\
&= \int_{\mathbb{R}^5} |u(\mathbf{x})|^{10/3} \ d\mu_5. \ \square
\end{aligned}$$

LEMMA 3.3.4 *Let* $w \in L^{10/7}(\mathbb{R}^5)$ *be cylindrically symmetric. Then* $\mathcal{K}w$ *is cylindrically symmetric and* $\mathcal{K}w \in E_c$.

Proof Let $\{\phi_n\}_{n=1}^\infty$ be a sequence of test functions converging to $\mathcal{K}w$ in E. Then $\{\phi_n\}_{n=1}^\infty$ is Cauchy and by Lemma 3.3.2 $\{\mathcal{A}\phi_n\}_{n=1}^\infty$ is Cauchy. Therefore $\mathcal{A}\phi_n \to f$ for some $f \in E_c$ and by (3.3.4) $\mathcal{A}\phi_n \to f$ in $L^{10/3}(\mathbb{R}^5)$. But Lemma 3.3.3 yields $\mathcal{A}\phi_n \to \mathcal{A}(\mathcal{K}w)$ in $L^{10/3}(\mathbb{R}^5)$ hence $\mathcal{A}(\mathcal{K}w) = f \in E_c$. Therefore $\mathcal{A}(\mathcal{K}w) \in E_c$ is cylindrically symmetric and

$$||\mathcal{A}(\mathcal{K}w)||_E = \lim_{n\to\infty} ||\mathcal{A}\phi_n||_E \leq ||\mathcal{K}w||_E. \tag{3.3.8}$$

Also, since w is cylindrically symmetric

$$\int_{\mathbb{R}^5} w\mathcal{A}(\mathcal{K}w)\ d\mu = \int_G \int_{\mathbb{R}^5} \mathcal{K}w(\theta\mathbf{x})w(\mathbf{x})\ d\mu d\theta = \int_{\mathbb{R}^5} \mathcal{K}w(\mathbf{x})w(\mathbf{x})\ d\mu. \qquad (3.3.9)$$

Combining (3.3.8) and (3.3.9) we obtain $\Phi_E^w(\mathcal{A}(\mathcal{K}w)) \leq \Phi_E^w(\mathcal{K}w)$ hence $\mathcal{K}w = \mathcal{A}(\mathcal{K}w)$. $\square$

LEMMA 3.3.5 *Let $v \in L^{10/7}(\Pi, \tau)$ be identified with $w \in L^{10/7}(\mathbb{R}^5)$. Then Kv is well-defined, $K : L^{10/7}(\Pi, \tau) \to H$ is a bounded linear operator and $Kv = r^2\mathcal{K}w$.*

Proof For $u \in H$ let $f = u/r^2$. From the isomorphism between H and E_c we have

$$\Phi_H^v(u) = \frac{1}{2}||u||_H^2 - \int_\Pi uv\ d\nu = \frac{1}{2}||f||_{E_c}^2 - \int_{\mathbb{R}^5} wf d\mu = \Phi_E^w(f) \quad \forall u \in H.$$

From Lemmas 3.3.1 and 3.3.4 it follows that Kv is well defined and $Kv = r^2\mathcal{K}w$. Also

$$||Kv||_H = ||\mathcal{K}w||_E \leq \frac{2k_1}{\pi}||w||_{10/7} = \frac{2k_1}{\pi^{3/10}}||v||_{10/7;\Pi;\tau}. \ \square$$

Recall

$$\Pi(\xi) = \{(r, z) \in \Pi | 0 < r < \xi\}.$$

For $v \in L^{10/7}(\Pi(\xi))$ we immediately obtain from Lemma 3.3.5 that Kv is well-defined and $K : L^{10/7}(\Pi(\xi)) \to H$ is bounded. We now show that this is true for $v \in L^p(\Pi(\xi))$ for $1 < p \leq 2$. We require a preliminary lemma.

For a function f defined on Π define the 'reflection' of f in the line $r = \xi$ to be

$$\bar{f}(r, z) = \begin{cases} f(r, z) & \text{if } 0 < r \leq \xi, \\ f(2\xi - r, z) & \text{if } \xi < r < 2\xi, \\ 0 & \text{otherwise.} \end{cases} \qquad (3.3.10)$$

LEMMA 3.3.6 *Let $u \in H$. Then $u \in W^{1,2}(\Pi(\xi))$ for each $\xi > 0$ and, in particular,*

$$||u||_{q;\Pi(\xi)} \leq M(\xi) \left(\int_{\Pi(\xi)} \frac{1}{r^2} |\nabla u|^2 d\nu \right)^{1/2}$$

where $2 \leq q < \infty$.

Proof By [20, Theorem 2], H is embedded in $L^2(\Pi, \sigma)$ where σ is the measure on Π with density $2\pi/r^3$ with respect to plane Lebesgue measure.

Let $\phi \in C_0^\infty(\Pi)$ and let $\bar{\phi}$ be the reflection of ϕ in $r = \xi$. Since $W^{1,\infty}(\Omega) = C^{0,1}(\Omega)$ for any bounded domain $\Omega \subset \mathbb{R}^2$ and $\bar{\phi}$ is Lipschitz we have $\bar{\phi} \in W^{1,2}(\Pi(2\xi))$ and, since the support of $\bar{\phi}$ is a compact subset of $\Pi(2\xi)$, $\bar{\phi} \in W_0^{1,2}(\Pi(2\xi))$.

We observe that

$$\begin{aligned}\int_{\Pi(2\xi)} |\nabla\bar{\phi}|^2 \, d\mu_2 &= 2\int_{\Pi(\xi)} |\nabla\phi|^2 \, d\mu_2 \\ &\leq \frac{\xi}{\pi}\int_{\Pi(\xi)} \frac{1}{r^2}|\nabla\phi|^2 \, d\nu.\end{aligned}$$

Also, by Poincaré's inequality $||\bar{\phi}||_{2;\Pi(2\xi)} \leq 2\xi||\nabla\bar{\phi}||_{2;\Pi(2\xi)}$. Hence

$$\begin{aligned}||\phi||^2_{1,2;\Pi(\xi)} &= ||\phi||^2_{2;\Pi(\xi)} + ||\nabla\phi||^2_{2;\Pi(\xi)} \\ &= \frac{1}{2}(||\bar{\phi}||^2_{2;\Pi(2\xi)} + ||\nabla\bar{\phi}||^2_{2;\Pi(2\xi)}) \\ &\leq \frac{1}{2}(4\xi^2+1)||\nabla\bar{\phi}||^2_{2;\Pi(2\xi)} \\ &\leq \frac{(4\xi^2+1)\xi}{2\pi}\left(\int_{\Pi(\xi)} \frac{1}{r^2}|\nabla\phi|^2 d\nu\right).\end{aligned}$$

The result follows from the embedding $W^{1,2}(\Pi(\xi)) \to L^q(\Pi(\xi))$, $2 \leq q < \infty$.

LEMMA 3.3.7 *For $v \in L^p(\Pi(\xi))$, $1 < p \leq 2$, Kv is well defined and $K : L^p(\Pi(\xi)) \to H$ is a bounded linear operator. Moreover, $Kv \in W^{2,p}_{loc}(\Pi)$, $\mathcal{L}(Kv) = v$ almost everywhere in Π and*

$$\int_\Pi vKv \, d\nu = \int_\Pi \frac{1}{r^2}|\nabla Kv|^2 \, d\nu = ||Kv||_H^2.$$

Proof Letting q denote the conjugate exponent of p we have from Lemma 3.3.6

$$\begin{aligned}\Phi_H^v(u) = \frac{1}{2}||u||_H^2 - \int_{\Pi(\xi)} uv \, d\nu &\geq \frac{1}{2}||u||_H^2 - ||v||_p\left(\int_{\Pi(\xi)} |u|^q d\nu\right)^{1/q} \\ &\geq \frac{1}{2}||u||_H^2 - M(\xi)||v||_p||u||_H\end{aligned}$$

$$\to \infty \quad \text{as } ||u||_H \to \infty.$$

Thus Φ_H^v is strictly convex, coercive and weakly lower semicontinuous on H hence there exists a unique minimiser, Kv, say.

The Fréchet derivative of the functional on H at Kv is identically zero. Hence

$$\int_\Pi \frac{1}{r^2}\nabla Kv.\nabla h \; d\nu = \int_\Pi hv \; d\nu \quad \forall h \in H \tag{3.3.11}$$

and, in particular,

$$\int_\Pi \frac{1}{r^2}\nabla Kv.\nabla \phi \; d\nu = \int_\Pi \phi v \; d\nu \quad \forall \phi \in C_0^\infty(\Pi). \tag{3.3.12}$$

Therefore

$$\int_\Pi Kv\mathcal{L}\phi \; d\nu = \int_\Pi \phi v \; d\nu \quad \forall \phi \in C_0^\infty(\Pi).$$

By Agmon [2, Theorem 6.1] $Kv \in W_{loc}^{2,p}(\Pi)$ and $\mathcal{L}(Kv) = v$ almost everywhere in Π.

From (3.3.11) we obtain

$$\int_\Pi \frac{1}{r^2}|\nabla Kv|^2 \; d\nu = \int_\Pi vKv \; d\nu.$$

Finally we prove that K is bounded. Considering $u = 0$ we have

$$\frac{1}{2}||Kv||_H^2 - \int_\Pi vKv \; d\nu \leq 0$$

hence

$$\frac{1}{2}||Kv||_H^2 \leq ||v||_p \left(\int_{\Pi(\xi)} |Kv|^q d\nu\right)^{1/q} \leq M(\xi)||v||_p||Kv||_H$$

where in the final inequality we have used Lemma 3.3.6. □

LEMMA 3.3.8 *Let $w \in L^p(\mathbb{R}^5) \cap L^{10/7}(\mathbb{R}^5)$ where $5/2 < p < \infty$. Then $\mathcal{K}w \in W_{loc}^{2,p}(\mathbb{R}^5) \cap C(\mathbb{R}^5)$ and $\mathcal{K}w(\mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$. If $p > 5$ then $\mathcal{K}w \in C^{1,\alpha}(\mathbb{R}^5)$ where $p(1-\alpha) \geq 5$.*

Proof We follow the method of Fraenkel and Berger [21, Lemma 3.1]. We recall from the proof of Lemma 3.3.1

$$\int_{\mathbb{R}^5} \nabla \mathcal{K}w . \nabla h \ d\mu = \int_{\mathbb{R}^5} wh \ d\mu \quad \forall h \in E.$$

Let $5/2 < p_1 < \min\{p, 10/3\}$ and let q_1 denote the conjugate exponent of p_1. Let $\mathbf{x} \in \mathbb{R}^5$ and for $j = 0, 1, 2, 3$ let $B_j = B_{R_j}(\mathbf{x})$ with $R_0 = 1/2$, $R_1 = 1$, $R_2 = 3/2$ and $R_3 = 2$. Then for any ball B in $\mathbb{R}^5$ and $\phi \in C_0^\infty(B)$ we have

$$\left| \int_B \mathcal{K}w \Delta\phi \ d\mu_5 \right| = \left| \int_B w\phi \ d\mu_5 \right| \le ||w||_{p_1;B} ||\phi||_{q_1;B} \ .$$

Applying [2, Theorem 6.1] yields $\mathcal{K}w \in W^{2,p_1}_{loc}(B_3)$ and

$$||\mathcal{K}w||_{2,p_1;B_1} \le M_1(||w||_{p_1;B_3} + ||\mathcal{K}w||_{p_1;B_2})$$

where M_1 is an absolute constant. Since $W^{2,p_1}(B_1)$ is embedded in $C(\overline{B_1})$ and therefore in $L^p(B_1)$ for all $p > 1$ we may apply Agmon's theorem again to obtain

$$\begin{aligned} ||\mathcal{K}w||_{2,p;B_0} &\le M(p)(||w||_{p;B_2} + ||\mathcal{K}w||_{p;B_1}) \\ &\le M(p)\left(||w||_{p;B_2} + c_1(p)M_1(||w||_{p_1;B_3} + ||\mathcal{K}w||_{p_1;B_2})\right) \end{aligned} \tag{3.3.13}$$

where $c_1(p)$ is the embedding constant associated with the embedding $W^{2,p_1}(B_1) \to L^p(B_1)$ and is independent of $\mathbf{x}$ (see [1, Lemma 5.17 and Definition 4.5]).

Also, by Lemma 3.3.1

$$||\mathcal{K}w||_{p_1;B_2} \le c_2(p_1)||\mathcal{K}w||_{10/3;B_2} \le c_2(p_1)||\mathcal{K}w||_{10/3} \le k_1 c_2(p_1)||\mathcal{K}w||_E \le \frac{2k_1^2 c_2(p_1)}{\pi}||w||_{10/7}.$$

Therefore

$$||\mathcal{K}w||_{2,p;B_0} \le M(p, p_1)\left(||w||_p + ||w||_{p_1} + ||w||_{10/7}\right)$$

where we note that the right hand side is independent of $\mathbf{x}$. Hence $\mathcal{K}w \in W^{2,p}_{loc}(\mathbb{R}^5)$. If $p > 5/2$ embedding theory yields

$$||\mathcal{K}w||_{C(\overline{B_0})} \le M(p)||\mathcal{K}w||_{2,p;B_0}$$

and if $p > 5$,

$$||\mathcal{K}w||_{C^{1,\alpha}(\overline{B_0})} \le M(p, \alpha)||\mathcal{K}w||_{2,p;B_0}$$

for $p(1-\alpha) \geq 5$. The results now follow immediately from (3.3.13). $\square$

LEMMA 3.3.9 *Let $w \in L^1(\mathbb{R}^5) \cap L^p(\mathbb{R}^5)$, $5/2 < p < \infty$. Then*

$$\mathcal{K}w(\mathbf{x}) = \frac{1}{8\pi^2} \int_{\mathbb{R}^5} \frac{1}{|\mathbf{x}-\mathbf{y}|^3} w(\mathbf{y}) d\mathbf{y}. \tag{3.3.14}$$

Proof Define

$$\mathcal{P}w(\mathbf{x}) = \frac{1}{8\pi^2} \int_{\mathbb{R}^5} \frac{1}{|\mathbf{x}-\mathbf{y}|^3} w(\mathbf{y}) d\mathbf{y} \tag{3.3.15}$$

to be the Newtonian potential of w.

Let $\mathcal{I}_\alpha$, $0 < \alpha < 5$, defined by

$$\mathcal{I}_\alpha(\mathbf{x}) = \frac{1}{\gamma(\alpha)|\mathbf{x}|^{5-\alpha}}$$

denote the Riesz kernel, where

$$\gamma(\alpha) = \frac{\pi^{5/2} 2^\alpha \Gamma(\alpha/2)}{\Gamma(5/2-\alpha/2)}.$$

The Riesz potential of a function f on $\mathbb{R}^5$ is defined as the convolution

$$\mathcal{I}_\alpha * f(\mathbf{x}) = \int_{\mathbb{R}^5} \mathcal{I}_\alpha(\mathbf{x}-\mathbf{y}) f(\mathbf{y}) d\mathbf{y} = \frac{1}{\gamma(\alpha)} \int_{\mathbb{R}^5} \frac{f(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{5-\alpha}} d\mathbf{y}.$$

Clearly $\mathcal{P}w = \mathcal{I}_2 * w$.

Let $1 < s < 5/2$ and let $s^* = 5s/(5-2s)$. By [38, Theorem 2.8.4]

$$||\mathcal{I}_2 * f||_{s^*} \leq M(s)||f||_s$$

for all $f \in L^s(\mathbb{R}^5)$. Hence $\mathcal{P} : L^s(\mathbb{R}^5) \to L^{s^*}(\mathbb{R}^5)$ is a bounded linear operator.

Let $w_n = w|_{B_n}$ where B_n is the ball of radius n centred at the origin. By [23, Theorem 9.9] $\mathcal{P}w_n \in W^{2,p}_{loc}(\mathbb{R}^5)$ and $-\Delta \mathcal{P}w_n = w_n$ for each n.

Let $\phi \in C_0^\infty(\mathbb{R}^5)$ with supp $\phi \subset B_{n_0}$. Then for $n > n_0$

$$\int_{\mathbb{R}^5} \mathcal{P}w_n(-\Delta\phi) = \int_{\mathbb{R}^5} w_n \phi$$

and letting $n \to \infty$,

$$\int_{\mathbb{R}^5} \mathcal{P}w(-\Delta\phi) = \int_{\mathbb{R}^5} w\phi.$$

Thus $-\Delta(\mathcal{P}w) = w$ in the distributional sense.

We show that $\mathcal{P}w(\mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$. Let $5/2 < p_1 \le p$ and $10/7 < p_2 < 5/2$ and let q_1, q_2 denote the corresponding conjugate exponents. Let $N > 1$ and $|\mathbf{x}| > 2N$. Letting $\rho = |\mathbf{x} - \mathbf{y}|$ we have

$$\begin{aligned}
8\pi^2\mathcal{P}w(\mathbf{x}) &= \int_{\mathbb{R}^5} \frac{1}{|\mathbf{x}-\mathbf{y}|^3} w(\mathbf{y})d\mathbf{y} \\
&= \int_{\rho<1} \frac{1}{\rho^3} w(\mathbf{y})d\mathbf{y} + \int_{\rho>N} \frac{1}{\rho^3} w(\mathbf{y})d\mathbf{y} + \int_{1\le\rho\le N} \frac{1}{\rho^3} w(\mathbf{y})d\mathbf{y} \\
&\le M_1 ||w|_{|\mathbf{x}|>N}||_{p_1} \left(\int_0^1 \rho^{4-3q_1} d\rho\right)^{1/q_1} + \frac{1}{N^3}||w||_1 \\
&\quad + M_2 ||w|_{|\mathbf{x}|>N}||_{p_2} \left(\int_1^N \rho^{4-3q_2} d\rho\right)^{1/q_2} \\
&\to 0 \qquad \text{as } N \to \infty
\end{aligned} \tag{3.3.16}$$

since $\int_0^1 \rho^{4-3q_1} d\rho$ and $\int_1^\infty \rho^{4-3q_2} d\rho$ are finite by the choice of p_1 and p_2.

Therefore by Lemmas 3.3.1 and 3.3.8

$$\begin{aligned}
-\Delta(\mathcal{K}w - \mathcal{P}w) &= 0 \ \text{ in } \mathbb{R}^5, \\
(\mathcal{K}w - \mathcal{P}w)(\mathbf{x}) &\to 0 \text{ as } |\mathbf{x}| \to \infty.
\end{aligned}$$

An application of the maximum principle yields $\mathcal{K}w = \mathcal{P}w$ almost everywhere. □

The Green's function for the operator $\mathcal{L}$ on Π (with ν measure) is

$$g(r, z, r', z') = \frac{rr'}{8\pi^2} \int_{-\pi}^{\pi} \frac{\cos\theta'}{(r^2 + r'^2 - 2rr'\cos\theta' + (z - z')^2)^{1/2}} d\theta'. \tag{3.3.17}$$

LEMMA 3.3.10 *Let* $5/2 < p < \infty$. *Let* $v \in L^1(\Pi, \tau) \cap L^p(\Pi, \tau)$ *and identify* v *with* $w \in L^1(\mathbb{R}^5) \cap L^p(\mathbb{R}^5)$. *Then*

$$r^2\mathcal{K}w(r, z) = Kv(r, z) = Tv(r, z) = \int_{\Pi} g(r, z, r', z')v(r', z') \ 2\pi r' dr' dz'.$$

Proof Firstly we note that Lemma 3.3.5 ensures Kv is well-defined and $Kv(r,z) = r^2\mathcal{K}w(r,z)$.

We recall $\mathcal{K}w$ is cylindrically symmetric. We shall make use of the cylindrical polar representation

$$\mathbf{y} = (r', \theta', \phi', \psi', z'),$$

where $r' \geq 0$, $0 \leq \theta', \phi' \leq \pi$, $-\pi \leq \psi' \leq \pi$, and

$$\begin{aligned}
y_1 &= r'\cos\theta' \\
y_2 &= r'\sin\theta'\cos\phi' \\
y_3 &= r'\sin\theta'\sin\phi'\cos\psi' \\
y_4 &= r'\sin\theta'\sin\phi'\sin\psi' \\
y_5 &= z'.
\end{aligned}$$

The volume element is $r'^3\sin^2\theta'\sin\phi'$. Similarly we make the transformation

$$\mathbf{x} = (r, \theta, \phi, \psi, z).$$

Applying this transformation to the representation of $\mathcal{K}w$ as the Newtonian potential of w and noting that $\mathcal{K}w(r,\theta,\phi,\psi,z) = \mathcal{K}w(r,0,0,0,z)$ for all θ we obtain

$$\begin{aligned}
&\mathcal{K}w(r,z) \\
= &\frac{1}{8\pi^2}\int_{-\infty}^{\infty}\int_0^{\infty}\int_{-\pi}^{\pi}\int_0^{\pi}\int_0^{\pi}\frac{w(r',z')\sin^2\theta'\sin\phi'}{(r^2+r'^2-2rr'\cos\theta'+(z-z')^2)^{3/2}}\ r'^3d\phi'd\theta'd\psi'dr'dz' \\
= &\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\pi}\frac{w(r',z')\sin^2\theta'}{(r^2+r'^2-2rr'\cos\theta'+(z-z')^2)^{3/2}}\ r'^3d\theta'dr'dz' \\
= &\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_0^{\infty}\left(\left[-\frac{\sin\theta'}{r(r^2+r'^2-2rr'\cos\theta'+(z-z')^2)^{1/2}}\right]_0^{\pi}\right. \\
&\left.+\int_0^{\pi}\frac{\cos\theta'}{r(r^2+r'^2-2rr'\cos\theta'+(z-z')^2)^{1/2}}d\theta'\right)w(r',z')\ r'^2dr'dz' \\
= &\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\pi}\frac{w(r',z')\cos\theta'}{r(r^2+r'^2-2rr'\cos\theta'+(z-z')^2)^{1/2}}\ r'^2d\theta'dr'dz' \\
= &\frac{1}{4\pi}\int_{-\infty}^{\infty}\int_0^{\infty}\int_{-\pi}^{\pi}\frac{w(r',z')\cos\theta'}{r(r^2+r'^2-2rr'\cos\theta'+(z-z')^2)^{1/2}}\ r'^2d\theta'dr'dz'
\end{aligned}$$

$$= \frac{1}{r^2}\int_{-\infty}^{\infty}\int_0^{\infty} g(r,z,r',z')w(r',z')\ 2\pi r'dr'dz'$$
$$= \frac{1}{r^2}Tv(r,z). \ \Box$$

3.4 Existence of maximisers

An application of [16, Theorem 3.1] provides the following result.

THEOREM 3.4.1 *Let $1 < p < \infty$. Let $U \subset \Pi$ be open, unbounded and have infinite ν-measure and let $f_0 \in L^p(U)$ be non-negative. Then*

$$\overline{R(f_0)}^w = \{f \geq 0, f \text{ measurable on } U | \int_U (f-\alpha)_+ d\nu \leq \int_U (f_0 - \alpha)_+ d\nu, \forall \alpha > 0\}.$$

Furthermore

(i) $\overline{R(f_0)}^w = \overline{conv\ R(f_0)}$,

(ii) The set of extreme points of $\overline{R(f_0)}^w$ is $RC(f_0)$,

(iii) $\overline{R(f_0)}^w$ is weakly compact.

LEMMA 3.4.2 *Let $1 < p < \infty$. Let $U \subset \Pi$ be open and let $f_0 \in L^1(U) \cap L^p(U)$. Then $\overline{R(f_0)}^w \subset L^1(U)$.*

Proof Let $f \in \overline{R(f_0)}^w$. Then there exists a sequence, $\{f_n\}_{n=1}^{\infty}$, of rearrangements of f_0, such that $f_n \xrightarrow{w} f$ in $L^p(U)$. Furthermore, there exists a sequence of convex combinations v_n, converging to f in $L^p(U)$ such that $v_n = \sum_{k=1}^{\infty} \alpha_{n,k} f_k$ where $\alpha_{n,k} \geq 0$, $\sum_{k=1}^{\infty} \alpha_{n,k} = 1$ for each n and only finitely many of the $\alpha_{n,k}$ are non-zero for each n. Passing to a subsequence if necessary we can assume $v_n \to f$ pointwise almost everywhere. Then

$$||v_n||_1 \leq ||\sum_{k=1}^{\infty} \alpha_{n,k} f_k||_1 \leq ||f_0||_1$$

and by Fatou's Lemma $||f||_1 \leq \liminf_{n\to\infty} ||v_n||_1 \leq ||f_0||_1$. $\Box$

3.4.1 Steiner-symmetrisation

Let $1 < p < \infty$ and let q denote the conjugate exponent of p. For non-negative $u \in L^p(\mathbb{R}^5)$ the Steiner-symmetrisation of u is defined similarly to that for functions defined on Π by rearranging the restriction of u to each line parallel to the z-axis as a symmetrically decreasing function.

For non-negative $u, v \in L^p(\mathbb{R}^5)$ and $w \in L^q(\mathbb{R}^5)$ the inequalities

$$\begin{aligned} \int_{\mathbb{R}^5} uw \; d\mu &\leq \int_{\mathbb{R}^5} u^* w^* \; d\mu && (3.4.18) \\ ||u^* - v^*||_p &\leq ||u - v||_p && (3.4.19) \end{aligned}$$

hold and the analogous inequalities with Π and ν in place of $\mathbb{R}^5$ and μ are also valid.

LEMMA 3.4.3 *Let $u \in E$. Then $u^+ \in E$ and, in particular, there exists a sequence of non-negative test functions converging to u^+ in E.*

Proof Let $u \in E$. We first show that if $\{\phi_n\}_{n=1}^{\infty}$ is a sequence of test functions converging to u in E then

$$\int_{\mathbb{R}^5} |\nabla \phi_n^+ - \nabla u^+|^2 d\mu \to 0$$

as $n \to \infty$.

Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \max\{x, 0\}$. Then f is Lipchitz and by [38, Lemma 2.1.11] for $g \in W^{1,2}_{loc}(\mathbb{R}^5)$ we have $\nabla(f \circ g) = f'(g)\nabla g$ almost everywhere.

Let $R = \{u < 0\}$, $Q = \{u = 0\}$ and $P = \{u > 0\}$. By [23, Theorem 7.7] $\nabla u = 0$ almost everywhere on Q hence

$$\begin{aligned} \int_Q |\nabla \phi_n^+ - \nabla u^+|^2 \; d\mu &= \int_Q |f'(\phi_n)\nabla\phi_n - f'(u)\nabla u|^2 \; d\mu \\ &= \int_Q |f'(\phi_n)\nabla\phi_n - f'(\phi_n)\nabla u|^2 \; d\mu \\ &\leq \int_Q |\nabla\phi_n - \nabla u|^2 \; d\mu \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Similarly, by the Dominated Convergence Theorem, it follows that

$$\int_R |\nabla \phi_n^+ - \nabla u^+|^2 \; d\mu \to 0 \quad \text{as } n \to \infty.$$

It remains to show that

$$\int_P |f'(\phi_n)\nabla\phi_n - f'(u)\nabla u|^2 \ d\mu \to 0 \quad \text{as } n \to \infty.$$

But

$$||f'(\phi_n)\nabla\phi_n - f'(u)\nabla u||_{2;P} \leq ||(f'(\phi_n) - f'(u))\nabla\phi_n||_{2;P} + ||f'(u)(\nabla\phi_n - \nabla u)||_{2;P}$$

and

$$||f'(u)(\nabla\phi_n - \nabla u)||_{2;P} \leq ||\nabla\phi_n - \nabla u||_{2;P} \to 0 \quad \text{as } n \to \infty.$$

Since $\phi_n \to u$ in $L^{10/3}(\mathbb{R}^5)$ there exists a subsequence ϕ_{n_k} such that $\phi_{n_k} \to u$ pointwise almost everywhere. By the Dominated Convergence Theorem

$$\begin{aligned} & ||f'(\phi_{n_k})\nabla\phi_{n_k} - f'(u)\nabla\phi_{n_k}||_{2;P} \\ \leq \ & ||f'(\phi_{n_k})(\nabla\phi_{n_k} - \nabla u)||_{2;P} + ||(f'(\phi_{n_k}) - f'(u))\nabla u||_{2:P} + ||f'(u)(\nabla u - \nabla\phi_{n_k})||_{2;P} \\ & \to 0 \quad \text{as } n_k \to \infty. \end{aligned}$$

Hence there exists a subsequence ϕ_{n_k} such that $\phi^+_{n_k} \to u^+$ in E. We note that $\phi^+_{n_k} \in W^{1,2}(\mathbb{R}^5)$ is non-negative . By [1, Lemma 3.15] there exists a sequence of non-negative test functions converging to $\phi^+_{n_k}$ in E and the result follows. □

LEMMA 3.4.4 *Let $u \in E$ be non-negative. Then*

$$||u^*||_E \leq ||u||_E.$$

Proof Let ϕ_n be a sequence of non-negative test functions converging to u in E. It is shown in [28] that for all functions $u \in W^{1,2}(\mathbb{R}^5)$ we have $u^* \in W^{1,2}(\mathbb{R}^5)$ and

$$\int_{\mathbb{R}^5} |\nabla u^*|^2 d\mu \leq \int_{\mathbb{R}^5} |\nabla u|^2 d\mu.$$

Hence ϕ_n^* is a Cauchy sequence in E which by (3.4.19) and (3.3.4) converges to u^* in E and the claim follows. □

LEMMA 3.4.5 *Let $u \in H$ be non-negative. Then*

$$||u^*||_H \leq ||u||_H.$$

Proof This follows immediately from Lemma 3.4.4 and the isomorphism between H and E_c. $\square$

3.4.2 Maximising sequences supported in a strip

LEMMA 3.4.6 *Let $2 < p < \infty$ and let q denote the conjugate exponent of p. Let $v \in L^p(\Pi)$ vanish outside a set of measure $2\pi^2 a^3$. Then*

$$Tv(r,z) \leq r\left(M_1(a,p)(\log r)^2 + M_2(a,p)(\log r) + M_3(a,p)\right)^{1/q} ||v||_p \quad \textit{if } r > e^{2/3}a.$$

Proof Friedman and Turkington [22, Lemma 3.3] derived the estimate

$$g(r,z,r',z') \leq \begin{cases} Cr\log(r/\rho) & \text{if } \rho \leq r/2, \\ Cr^2r'^2/\rho^3 & \text{if } \rho \geq r/2 \end{cases} \tag{3.4.20}$$

where $\rho = ((r-r')^2 + (z-z')^2)^{1/2}$ and C is an absolute constant.

If $\rho \geq r/2$ then $r' \leq r + \rho \leq 3\rho$ and

$$\begin{aligned} \int_{\rho \geq r/2} g(r,z,r',z')v(r',z')\ 2\pi r'dr'dz' &\leq \int_{\rho\geq r/2} \frac{9Cr^2}{\rho} v(r',z')\ 2\pi r'dr'dz' \\ &\leq 18Cr||v||_1. \end{aligned}$$

For $\mathbf{x} = (r,z) \in \Pi$ with $r > a$ and $\mathbf{y} = (r',z') \in \Pi$ we can define $\hat{v}_{\mathbf{x}}(\mathbf{y})$ as the rearrangement of v (with respect to ν-measure) as a decreasing function of ρ only. Then $\hat{v}_{\mathbf{x}}(\mathbf{y})$ is positive when $\rho < (a^3/r)^{1/2}$ and, in particular, by Hölder's inequality we have

$$||v||_1 = ||\hat{v}_{\mathbf{x}}||_1 \leq (2\pi^2 a^3)^{1/q}||v||_p.$$

Hence

$$\int_{\rho\geq r/2} g(r,z,r',z')v(r',z')\ 2\pi r'dr'dz' \leq M_1(a,p)r||v||_p. \tag{3.4.21}$$

If $r/2 > (a^3/r)^{1/2}$ then

$$\begin{aligned}
\int_{\rho\leq r/2} g(r,z,r',z')v(r',z')\ 2\pi r'dr'dz' &\leq \int_{\rho\leq r/2} Cr\log(\frac{r}{\rho})v(r',z')\ 2\pi r'dr'dz' \\
&\leq \int_{\rho\leq(a^3/r)^{1/2}} Cr\log(\frac{r}{\rho})\hat{v}_{\mathbf{x}}(r',z')\ 2\pi r'dr'dz' \\
&\leq Cr\left(4\pi^2 r\int_0^{(a^3/r)^{1/2}} \left|\log\left(\frac{\rho}{r}\right)\right|^q \rho d\rho\right)^{1/q} ||\hat{v}_{\mathbf{x}}||_p \\
&= Cr(2\pi)^{2/q}\left(r^3\int_0^{(a/r)^{3/2}} |\log u|^q\, udu\right)^{1/q} ||v||_p.
\end{aligned}$$

But if $\log\alpha < -1$

$$\begin{aligned}
\int_0^{\alpha} |\log u|^q u\ du &\leq \int_{-\infty}^{\log\alpha} e^{2z}z^2\ dz \\
&= \frac{\alpha^2}{2}\left((\log\alpha)^2 - \log\alpha + \frac{1}{2}\right) \\
&\leq \frac{\alpha^2}{2}\left(2(\log\alpha)^2 + \frac{1}{2}\right).
\end{aligned}$$

Therefore if $r > e^{2/3}a$

$$\int_{\rho\leq r/2} g(r,z,r',z')v(r',z')\ 2\pi r'dr'dz' \leq r\left(M_2(a)(\log r)^2 + M_3(a)\log r + M_4(a)\right)^{1/q} ||v||_p. \tag{3.4.22}$$

Combining (3.4.21) and (3.4.22) we have

$$\begin{aligned}
Tv(r,z) &= \int_\Pi g(r,z,r',z')v(r',z')\ 2\pi r'dr'dz' \\
&\leq r\left[M_1(a,p) + (M_2(a)(\log r)^2 + M_3(a)\log r + M_4(a))^{1/q}\right] ||v||_p \\
&\leq r\left[2^{1-\frac{1}{q}}\left(M_1(a,p)^q + M_2(a)(\log r)^2 + M_3(a)\log r + M_4(a)\right)^{1/q}\right] ||v||_p
\end{aligned}$$

if $r > e^{2/3}a$ where we have used the concavity of $s \mapsto s^{1/q}$. $\square$

LEMMA 3.4.7 *Let $2 < p < \infty$ and let q denote the conjugate exponent of p.*

Let $\zeta_0 \in L^p(\Pi)$ be non-negative and vanish outside a set of measure $2\pi^2 a^3$. Then for all $\zeta \in \overline{R(\zeta_0)}^w$

$$T\zeta(r,z) \le r\left(M_1(a,p)(\log r)^2 + M_2(a,p)\log r + M_3(a,p)\right)^{1/q} ||\zeta_0||_p \quad \textit{if } r > e^{2/3}a.$$

Proof Let $v \in R(\zeta_0)$ and let $v_\xi = v|_{\Pi(\xi)}$. Then by Lemma 3.4.6

$$Tv_\xi(r,z) \le r\left(M_1(a,p)(\log r)^2 + M_2(a,p)\log r + M_3(a,p)\right)^{1/q} ||\zeta_0||_p \quad (3.4.23)$$

if $r > e^{2/3}a$. Now consider $\zeta \in \overline{R(\zeta_0)}^w$ and let $\zeta_\xi = \zeta|_{\Pi(\xi)}$. By Theorem 3.4.1 the weak closure of $R(\zeta_0)$ in $L^p(\Pi)$ is the same as the weak closure of $R(\zeta_0)$ in $L^2(\Pi)$. Hence there exists a sequence, f_n, of rearrangements of ζ_0 such that $f_n \xrightarrow{w} \zeta$ in $L^2(\Pi)$. Furthermore, by Mazur's lemma there exists a sequence of convex combinations, v_n, converging strongly to ζ in $L^2(\Pi)$ such that $v_n = \sum_{k=1}^{\infty} c_{n,k} f_k$ where $c_{n,k} \ge 0$, $\sum_{k=1}^{\infty} c_{n,k} = 1$ for each n and only finitely many of the $c_{n,k}$ are non-zero for each n. In particular $v_n|_{\Pi(\xi)} =: v_{n,\xi} \to \zeta_\xi$ in $L^2(\Pi(\xi))$. But from Lemmas 3.3.6 and 3.3.7 it follows that $T : L^2(\Pi(\xi)) \to L^2(\Pi(\xi))$ is bounded hence $Tv_{n,\xi} \to T\zeta_\xi$ in $L^2(\Pi(\xi))$ as $n \to \infty$. By (3.4.23)

$$\begin{aligned}
Tv_{n,\xi}(r,z) &= T\left(\sum_{k=1}^{\infty} c_{n,k} f_k|_{\Pi(\xi)}\right)(r,z) \\
&= \sum_{k=1}^{\infty} c_{n,k} T(f_k|_{\Pi(\xi)})(r,z) \\
&\le r\left(M_1(a,p)(\log r)^2 + M_2(a,p)\log r + M_3(a,p)\right)^{1/q} ||\zeta_0||_p \quad \text{if } r > e^{2/3}a.
\end{aligned}$$

Also there exists a subsequence $Tv_{n,\xi}$ converging to $T\zeta_\xi$ pointwise almost everywhere in $\Pi(\xi)$ so that

$$T\zeta_\xi(r,z) \le r\left(M_1(a,p)(\log r)^2 + M_2(a,p)\log r + M_3(a,p)\right)^{1/q} ||\zeta_0||_p$$

if $e^{2/3}a < r < \xi$. Applying the Monotone Convergence Theorem we obtain

$$T\zeta(r,z) = \int_\Pi g(r,z,r',z')\zeta(r',z')\ 2\pi r' dr' dz'$$

$$
\begin{aligned}
&= \lim_{n\to\infty}\int_\Pi g(r,z,r',z')\zeta_n(r',z')\ 2\pi r'dr'dz' \\
&\leq r\left(M_1(a,p)(\log r)^2 + M_2(a,p)\log r + M_3(a,p)\right)^{1/q}||\zeta_0||_p \text{ if } r > e^{2/3}a. \ \square
\end{aligned}
$$

We recall that for $\lambda > 0$,

$$
\Psi_\lambda(\zeta) = \frac{1}{2}\int_\Pi \zeta T\zeta d\nu - \frac{\lambda}{2}\int_\Pi r^2\zeta d\nu. \tag{3.4.24}
$$

LEMMA 3.4.8 *Let $2 < p < \infty$. Let $\zeta_0 \in L^p(\Pi)$ be non-negative and vanish outside a set of finite measure β. Then there exists $\xi(\lambda,\beta)$ and a Steiner-symmetric maximising sequence, $\{\zeta_n^*\}_{n=1}^\infty$, for Ψ_λ relative to $\overline{R(\zeta_0)_b}^w$ with supp $\zeta_n^* \subset \Pi(\xi(\lambda,\beta))$.*

Furthermore, if Ψ_λ attains a maximum relative to $\overline{R(\zeta_0)_b}^w$ and $\tilde\zeta_\lambda$ is a maximiser, then

$$
\tilde\zeta_\lambda^{-1}(0,\infty) \subset (T\tilde\zeta_\lambda - \frac{1}{2}\lambda r^2)^{-1}[0,\infty) \subset \Pi(\xi(\lambda,\beta))
$$

except for sets of zero measure.

Proof Lemma 3.4.7 ensures that $\Psi_\lambda(\zeta)$ is finite for all $\zeta \in \overline{R(\zeta_0)_b}^w$.

Let $\{\zeta_n\}_{n=1}^\infty$ be a maximising sequence for Ψ_λ relative to $\overline{R(\zeta_0)_b}^w$. Let

$$
\begin{aligned}
A_n^+ &= \{(r,z) \in \Pi | T\zeta_n(r,z) - \frac{1}{2}\lambda r^2 \geq 0\}, \\
A_n^- &= \{(r,z) \in \Pi | T\zeta_n(r,z) - \frac{1}{2}\lambda r^2 < 0\}
\end{aligned}
$$

and write $\zeta_n = \zeta_{n,1} + \zeta_{n,2}$ where $\zeta_{n,1} = \zeta_n 1_{A_n^+}$ and $\zeta_{n,2} = \zeta_n 1_{A_n^-}$. Then by Theorem 3.4.1 $\zeta_{n,1}, \zeta_{n,2} \in \overline{R(\zeta_0)_b}^w$ for all n.

By Lemma 3.4.7 there exists $\xi(\lambda,\beta)$ such that

$$
T\zeta(r,z) - \frac{1}{2}\lambda r^2 < 0 \quad \text{if } r \geq \xi(\lambda,\beta), \quad \text{for all } \zeta \in \overline{R(\zeta_0)}^w, \tag{3.4.25}
$$

hence $A_n^+ \subset \Pi(\xi(\lambda,\beta))$. Also

$$
\begin{aligned}
\Psi_\lambda(\zeta_n) - \Psi_\lambda(\zeta_{n,1}) &= \frac{1}{2}\int_\Pi(\zeta_n T\zeta_n - \zeta_{n,1}T\zeta_{n,1})\ d\nu - \frac{\lambda}{2}\int_\Pi r^2(\zeta_n - \zeta_{n,1})\ d\nu \\
&= \frac{1}{2}\int_\Pi(\zeta_{n,1}T\zeta_{n,2} + \zeta_{n,2}T\zeta_n)\ d\nu - \frac{\lambda}{2}\int_\Pi r^2\zeta_{n,2}\ d\nu
\end{aligned}
$$

$$\leq \int_\Pi \zeta_{n,2}(T\zeta_n - \frac{1}{2}\lambda r^2)\ d\nu \leq 0 \tag{3.4.26}$$

for all n. Hence $\zeta_{n,1}$ forms a maximising sequence for Ψ_λ relative to $\overline{R(\zeta_0)_b}^w$.

Let $\zeta^*_{n,1}$ denote the Steiner-symmetrisation of $\zeta_{n,1}$. Then $\zeta^*_{n,1} \in \overline{R(\zeta_0)_b}^w$ for all n. We observe that $r^2\zeta_{n,1}$ and $r^2\zeta^*_{n,1}$ are rearrangements of each other hence

$$\int_\Pi r^2\zeta_{n,1}\ d\nu = \int_\Pi r^2\zeta^*_{n,1}\ d\nu.$$

Also, by Riesz's inequality and the formula (3.3.17)

$$\int_\Pi \zeta_{n,1}T\zeta_{n,1}\ d\nu \leq \int_\Pi \zeta^*_{n,1}T\zeta^*_{n,1}\ d\nu$$

thus $\Psi_\lambda(\zeta_{n,1}) \leq \Psi_\lambda(\zeta^*_{n,1})$ for all n and $\{\zeta^*_{n,1}\}_{n=1}^\infty$ forms a maximising sequence for Ψ_λ with the required properties.

Suppose Ψ_λ attains a maximum relative to $\overline{R(\zeta_0)_b}^w$ and $\tilde{\zeta}_\lambda$ is a maximiser. Then if $\tilde{\zeta}_\lambda$ is not zero almost everywhere on $\{T\tilde{\zeta}_\lambda - \frac{1}{2}\lambda r^2 < 0\}$ there is strict inequality in (3.4.26) which is a contradiction. □

For $1 < p < \infty$ and $\xi > 0$ let

$$\begin{aligned} S(\Pi(\xi))^+_p &= \{v \in L^p(\Pi(\xi)) | v \in L^1(\Pi(\xi)), v \geq 0, v \text{ symmetric decreasing in } z\}, \\ U(\Pi(\xi)) &= \{u \geq 0 | u \in H^1_0(\Pi(\xi)), u \text{ symmetric decreasing in } z\}. \end{aligned}$$

Let H^* denote the non-negative Steiner-symmetric functions in H.

LEMMA 3.4.9 *Let $10/7 < p < 2$. Define $F : S(\Pi(\xi))^+_p \to \mathbb{R}$ by*

$$F(\zeta) = \int_\Pi \zeta K\zeta d\nu.$$

Then F is weakly sequentially continuous.

Proof Let $\zeta \in S(\Pi(\xi))^+_p$. Identify ζ with a non-negative Steiner-symmetric function $w \in L^1(\mathbb{R}^5) \cap L^p(\mathbb{R}^5)$. By Lemma 3.3.5 and the maximum principle, $r^2\mathcal{K}w = K\zeta \geq 0$.

We show $K\zeta$ is Steiner-symmetric. Recall that $K\zeta$ is the unique minimiser over $u \in H$ of the functional

$$\Phi_H^\zeta(u) = \frac{1}{2}||u||_H^2 - \int_\Pi u\zeta d\nu.$$

Let $u = K\zeta$. Then by Lemma 3.4.5

$$\begin{aligned}
\Phi_H^\zeta(u^*) &= \frac{1}{2}||u^*||_H^2 - \int_\Pi u^*\zeta d\nu \\
&\leq \frac{1}{2}||u||_H^2 - \int_\Pi u\zeta d\nu \\
&= \Phi_H^\zeta(u)
\end{aligned}$$

and since the minimiser is unique $(K\zeta)^* = K\zeta$ hence $K\zeta \in H^*$.

By [29, Theorem III.2] $U(\Pi(2\xi))$ is compactly embedded in $L^q(\Pi(2\xi), \mu_2)$ where q denotes the conjugate exponent of p.

Define $\Xi : H^* \to U(\Pi(2\xi))$ by $\Xi u = \bar{u}$ where $\bar{u}$ is the reflection of u in $r = \xi$ as defined by (3.3.10). Then Ξ is well defined by the calculations in Lemma 3.3.6 and for $u \in H^*$

$$\begin{aligned}
||\Xi u||_{H_0^1(\Pi(2\xi))}^2 = \int_{\Pi(2\xi)} |\nabla\bar{u}|^2 d\mu_2 &= 2\int_{\Pi(\xi)} |\nabla u|^2 d\mu_2 \\
&\leq \frac{\xi}{\pi}\int_{\Pi(\xi)} \frac{1}{r^2}|\nabla u|^2 d\nu \\
&\leq \frac{\xi}{\pi}||u||_H^2. \qquad (3.4.27)
\end{aligned}$$

Also for $u \in L^q(\Pi(2\xi), \mu_2)$

$$\int_{\Pi(\xi)} |u|^q d\nu \leq 2\pi\xi \int_{\Pi(\xi)} |u|^q d\mu_2 \leq 2\pi\xi \int_{\Pi(2\xi)} |u|^q d\mu_2. \qquad (3.4.28)$$

Therefore by Lemma 3.3.7, (3.4.27) and (3.4.28) we have

$$\begin{aligned}
K : S(\Pi(\xi))_p^+ &\to H^* \text{ is bounded,} \\
\Xi : H^* &\to U(\Pi(2\xi)) \text{ is bounded,}
\end{aligned}$$

the compact embedding

$$U(\Pi(2\xi)) \to L^q(\Pi(2\xi), \mu_2)$$

and the embedding

$$L^q(\Pi(2\xi), \mu_2) \to L^q(\Pi(\xi)).$$

We conclude $K : S(\Pi(\xi))_p^+ \to L^q(\Pi(\xi))$ is compact.

Let $\{v_n\}_{n=1}^\infty \in S(\Pi(\xi))_p^+$ and $v_0 \in S(\Pi(\xi))_p^+$ be such that $v_n \xrightarrow{w} v_0$ in $L^p(\Pi(\xi))$. Then

$$\left| \int_{\Pi(\xi)} (v_0 - v_n)(Kv_0 - Kv_n) \; d\nu \right| \to 0 \quad \text{as } n \to \infty.$$

Therefore

$$\int_{\Pi(\xi)} v_0 K v_0 \; d\nu + \lim_{n\to\infty} \int_{\Pi(\xi)} v_n K v_n \; d\nu - \lim_{n\to\infty} \left(\int_{\Pi(\xi)} v_n K v_0 + \int_{\Pi(\xi)} v_0 K v_n \; d\nu \right) = 0$$

and

$$\lim_{n\to\infty} \int_{\Pi(\xi)} v_n K v_n \; d\nu = \int_{\Pi(\xi)} v_0 K v_0 \; d\nu$$

since $Kv_0 \in L^q(\Pi(\xi))$ and $K : S(\Pi(\xi))_p^+ \to L^q(\Pi(\xi))$ is compact. Thus F is weakly sequentially continuous. □

LEMMA 3.4.10 *Let $1 \le p < \infty$. Then for non-negative $v \in L^p(\Pi)$, the map $v \mapsto \int_\Pi r^2 v \; d\nu$ is weakly lower semicontinuous.*

Proof This is immediate from properties of integral functionals given, for example, in [17]. □

We note that for $1 < t < p < \infty$ the weak closure of $R(\zeta_0)$ in $L^t(\Pi)$ is the same as the weak closure of $R(\zeta_0)$ in $L^p(\Pi)$. Let $R_\xi(\zeta_0)$ be the set of rearrangements of ζ_0 supported on $\Pi(\xi)$ and let $\overline{R_\xi(\zeta_0)}^w$ denote the weak closure of $R_\xi(\zeta_0)$ in $L^p(\Pi(\xi))$. $\overline{R_\xi(\zeta_0)}^w$ comprises those functions in $\overline{R(\zeta_0)}^w$ that are supported on $\Pi(\xi)$.

The remaining results of this section and those of 3.4.3 are modifications of the results in [15, Chapter 7].

THEOREM 3.4.11 *Let $5/2 < p < \infty$. Let $\zeta_0 \in L^p(\Pi)$ be non-negative and vanish outside a set of finite measure. Then for each $\lambda > 0$, Ψ_λ attains a maximum relative to $\overline{R_\xi(\zeta_0)}^w$ and all such maximisers belong to $RC(\zeta_0)$.*

Proof For $\lambda > 0$ let $\{\zeta_n\}_{n=1}^\infty$ be a Steiner-symmetric maximising sequence for Ψ_λ relative to $\overline{R_\xi(\zeta_0)}^w$. Such a sequence exists by Riesz's inequality and the methods of Lemma 3.4.8.

Let $10/7 < s < 2$. Then $\{\zeta_n\}_{n=1}^{\infty}$ is also a maximising sequence for Ψ_λ relative to the weak closure of $R(\zeta_0)$ in $L^s(\Pi(\xi))$. By Lemma 3.4.2, $\zeta_n \in S(\Pi(\xi))_s^+$ for all n.

Let $\overline{R}^*$ denote the elements of $\overline{R_\xi(\zeta_0)}^w$ that are Steiner-symmetric (or equivalently those Steiner-symmetric functions in $\overline{R(\zeta_0)}^w$ that are supported on $\Pi(\xi)$). Then $\overline{R}^*$ is convex and closed in $L^s(\Pi(\xi))$ (by the non-expansiveness of Steiner-symmetrisation). Hence $\overline{R}^*$ is weakly closed in $L^s(\Pi(\xi))$. By weak compactness, passing to a subsequence if necessary $\zeta_n \xrightarrow{w} \tilde{\zeta} \in \overline{R}^*$ in $L^s(\Pi(\xi))$. Then

$$\begin{aligned}
\Psi_\lambda(\tilde{\zeta}) &= \frac{1}{2}\int_{\Pi(\xi)} \tilde{\zeta} T\tilde{\zeta}\ d\nu - \frac{\lambda}{2}\int_{\Pi(\xi)} r^2\tilde{\zeta}\ d\nu \\
&= \frac{1}{2}\int_{\Pi(\xi)} \tilde{\zeta} \mathcal{K}\tilde{\zeta}\ d\nu - \frac{\lambda}{2}\int_{\Pi(\xi)} r^2\tilde{\zeta}\ d\nu \\
&\geq \lim_{n\to\infty} \frac{1}{2}\int_{\Pi(\xi)} \zeta_n \mathcal{K}\zeta_n\ d\nu - \frac{\lambda}{2}\liminf_{n\to\infty}\int_{\Pi(\xi)} r^2\zeta_n\ d\nu \\
&= \limsup_{n\to\infty} \frac{1}{2}\int_{\Pi(\xi)} \zeta_n \mathcal{K}\zeta_n\ d\nu + \limsup_{n\to\infty}\{-\frac{\lambda}{2}\int_{\Pi(\xi)} r^2\zeta_n\ d\nu\} \\
&\geq \limsup_{n\to\infty} \Psi_\lambda(\zeta_n) \\
&\geq \sup_{\zeta\in\overline{R_\xi(\zeta_0)}^w} \Psi_\lambda(\zeta).
\end{aligned}$$

Therefore Ψ_λ attains a maximum relative to $\overline{R_\xi(\zeta_0)}^w$.

Since Ψ_λ is strictly convex on $L^s(\Pi(\xi))$ any maximiser of Ψ_λ relative to $\overline{R_\xi(\zeta_0)}^w$ is an extreme point of $\overline{R_\xi(\zeta_0)}^w$ and by Theorem 3.4.1 the maximiser is a rearrangement of a curtailment of ζ_0. □

LEMMA 3.4.12 *Let* $5/2 < p < \infty$. *Let* $v \in L^1(\Pi(\xi)) \cap L^p(\Pi(\xi))$. *Then* $Tv - \frac{1}{2}\lambda r^2 \in L^\infty(\Pi(\xi))$.

Proof Identify v with $w \in L^1(\mathbb{R}^5) \cap L^p(\mathbb{R}^5)$. By Lemmas 3.3.8 and 3.3.10 $Tv = r^2\mathcal{K}w$, $\mathcal{K}w$ is continuous and $\mathcal{K}w(\mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$. Hence $\mathcal{K}w \in L^\infty(\mathbb{R}^5)$ and therefore $Tv \in L^\infty(\Pi(\xi))$. Clearly $\frac{1}{2}\lambda r^2 \in L^\infty(\Pi(\xi))$. □

LEMMA 3.4.13 *Let* $5/2 < p < \infty$. *Let* $\zeta_0 \in L^p(\Pi)$ *be non-negative and vanish outside a set of finite measure. For* $\lambda > 0$ *let* $\tilde{\zeta}_\lambda$ *be a maximiser of* Ψ_λ *relative to* $\overline{R_\xi(\zeta_0)}^w$. *Then*

$$\tilde{\zeta}_\lambda^{-1}(0,\infty) \subset (T\tilde{\zeta}_\lambda - \frac{1}{2}\lambda r^2)^{-1}(0,\infty) \tag{3.4.29}$$

except for a set of measure zero and moreover if $\tilde{\zeta}_\lambda \notin R(\zeta_0)$

$$\tilde{\zeta}_\lambda^{-1}(0,\infty) = (T\tilde{\zeta}_\lambda - \frac{1}{2}\lambda r^2)^{-1}(0,\infty) \tag{3.4.30}$$

except for a set of measure zero.

Proof We work in the space $L^1(\Pi(\xi))\cap L^p(\Pi(\xi))$ with the norm $||\cdot|| = ||\cdot||_1+||\cdot||_p$. For $v, h \in L^1(\Pi(\xi)) \cap L^p(\Pi(\xi))$

$$\Psi_\lambda(v+h) - \Psi_\lambda(v) = \int_{\Pi(\xi)} hTv\ d\nu + \frac{1}{2}\int_{\Pi(\xi)} hTh\ d\nu - \frac{\lambda}{2}\int_{\Pi(\xi)} r^2 h\ d\nu \tag{3.4.31}$$

and since $Tv - \frac{1}{2}\lambda r^2 \in L^\infty(\Pi(\xi))$ (and therefore in the dual space of $L^1(\Pi(\xi)) \cap L^p(\Pi(\xi))$ we deduce that Ψ_λ is subdifferentiable and $Tv - \frac{1}{2}\lambda r^2 \in \partial\Psi_\lambda(v)$.

Since Ψ_λ is strictly convex on $L^1(\Pi(\xi)) \cap L^p(\Pi(\xi))$

$$\Psi_\lambda(\tilde{\zeta}_\lambda) \geq \Psi_\lambda(\zeta) > \int_{\Pi(\xi)} (T\tilde{\zeta}_\lambda - \frac{1}{2}\lambda r^2)(\zeta - \tilde{\zeta}_\lambda) d\nu + \Psi_\lambda(\tilde{\zeta}_\lambda)$$

for all $\zeta \in \overline{R_\xi(\zeta_0)}^w \backslash \{\tilde{\zeta}_\lambda\}$ hence

$$\int_{\Pi(\xi)} (T\tilde{\zeta}_\lambda - \frac{1}{2}\lambda r^2)\zeta\ d\nu < \int_{\Pi(\xi)} (T\tilde{\zeta}_\lambda - \frac{1}{2}\lambda r^2)\tilde{\zeta}_\lambda\ d\nu \tag{3.4.32}$$

for all $\zeta \in \overline{R_\xi(\zeta_0)}^w \backslash \{\tilde{\zeta}_\lambda\}$.

Suppose, for a contradiction, (3.4.29) is false. Then there exists a set of positive measure $A \subset \tilde{\zeta}_\lambda^{-1}(0,\infty)$ such that $(T\tilde{\zeta}_\lambda - \frac{1}{2}\lambda r^2) \leq 0$ on A. For $\mathbf{x} \in \Pi$ define

$$\bar{\zeta}(\mathbf{x}) = \begin{cases} \tilde{\zeta}_\lambda(\mathbf{x}) & \mathbf{x} \in \Pi \backslash A \\ 0 & \mathbf{x} \in A \end{cases}$$

Then $\bar{\zeta} \in \overline{R_\xi(\zeta_0)}^w \backslash \tilde{\zeta}_\lambda$ and

$$\int_{\Pi(\xi)} (T\tilde{\zeta}_\lambda - \frac{1}{2}\lambda r^2)\bar{\zeta}\ d\nu \geq \int_{\Pi(\xi)} (T\tilde{\zeta}_\lambda - \frac{1}{2}\lambda r^2)\tilde{\zeta}_\lambda\ d\nu$$

which contradicts (3.4.32). This completes the proof of (3.4.29).

The proof of (3.4.30) is similar to that of [15, Chapter 7, Lemma 5]. □

THEOREM 3.4.14 *Let $5/2 < p < \infty$. Let $\zeta_0 \in L^p(\Pi)$ be non-negative and* *vanish outside a set of finite measure β. For $\lambda > 0$, let $\xi = \xi(\lambda, \beta)$ be such that for all $\zeta \in \overline{R(\zeta_0)}^w$,*

$$T\zeta(r, z) - \frac{1}{2}\lambda r^2 < 0 \quad \text{if } r \geq \xi(\lambda, \beta).$$

Then $\tilde{\zeta}_\lambda$ (which is in $RC(\zeta_0)$ by Theorem 3.4.11) maximises Ψ_λ relative to $\overline{R_\xi(\zeta_0)}^w$ if and only if $\tilde{\zeta}_\lambda$ maximises Ψ_λ relative to $\overline{R(\zeta_0)}_b^w$.

Proof By the proof of Lemma 3.4.8

$$\Psi_\lambda(\zeta) \leq \Psi_\lambda(\zeta|_{\Pi(\xi)}) \text{ for all } \zeta \in \overline{R(\zeta_0)}_b^w$$

with strict inequality if ζ is not zero almost everywhere outside $\Pi(\xi)$. Let $\tilde{\zeta}$ maximise Ψ_λ relative to $\overline{R_\xi(\zeta_0)}^w$. We show that $\tilde{\zeta}$ has bounded support and therefore $\tilde{\zeta}$ maximises Ψ_λ relative to $\overline{R(\zeta_0)}_b^w$.

Identify $\tilde{\zeta}$ with $w \in L^1(\mathbb{R}^5) \cap L^p(\mathbb{R}^5)$. By Lemmas 3.3.8 and 3.3.10, $T\tilde{\zeta} = r^2\mathcal{K}w$ and $\mathcal{K}w(\mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$. Hence $(T\tilde{\zeta} - \frac{1}{2}\lambda r^2)^{-1}(0, \infty)$ is a bounded set and by Lemma 3.4.13, $\tilde{\zeta}^{-1}(0, \infty)$ is a bounded subset of $\Pi(\xi)$ except for a set of zero measure. □

3.4.3 The existence of weak solutions

LEMMA 3.4.15 *Let $5/2 < p < \infty$. Let $\zeta_0 \in L^p(\Pi)$ be non-negative and vanish outside a set of finite measure β. For $\lambda > 0$ let $\xi = \xi(\lambda, \beta)$ be such that for all $\zeta \in \overline{R(\zeta_0)}^w$,*

$$T\zeta(r, z) - \frac{1}{2}\lambda r^2 < 0 \quad \text{if } r \geq \xi(\lambda, \beta).$$

Let $\tilde{\zeta}_\lambda$ be a maximiser of Ψ_λ relative to $\overline{R_\xi(\zeta_0)}^w$ and suppose $\tilde{\zeta}_\lambda \in R(\zeta_0)$. Then $\tilde{u} = T\tilde{\zeta}_\lambda$ satisfies

$$\mathcal{L}\tilde{u} = \phi \circ (\tilde{u} - \frac{1}{2}\lambda r^2)$$

almost everywhere in Π, for some increasing function ϕ.

Proof By the proof of Lemma 3.4.13, Ψ_λ is strictly convex on $L^1(\Pi(\xi)) \cap L^p(\Pi(\xi))$ and

$$\Psi_\lambda(\tilde{\zeta}_\lambda) \geq \Psi_\lambda(\zeta) > \int_{\Pi(\xi)} (T\tilde{\zeta}_\lambda - \frac{1}{2}\lambda r^2)(\zeta - \tilde{\zeta}_\lambda) d\nu + \Psi_\lambda(\tilde{\zeta}_\lambda)$$

for all $\zeta \in R(\zeta_0)\backslash\{\tilde{\zeta}_\lambda\}$ supported on $\Pi(\xi)$ which yields

$$\int_{\Pi(\xi)} (T\tilde{\zeta}_\lambda - \frac{1}{2}\lambda r^2)\zeta \, d\nu < \int_{\Pi(\xi)} (T\tilde{\zeta}_\lambda - \frac{1}{2}\lambda r^2)\tilde{\zeta}_\lambda \, d\nu$$

for all $\zeta \in R(\zeta_0)\backslash\{\tilde{\zeta}_\lambda\}$ supported on $\Pi(\xi)$.

By [15, Chapter 7, Theorem 1] $\tilde{\zeta}_\lambda = \phi \circ (T\tilde{\zeta}_\lambda - \frac{1}{2}\lambda r^2)$ almost everywhere on $\Pi(\xi)$ for some increasing function ϕ and we can assume that $\phi(s) \geq 0$ for all $s \in \text{dom } \phi$. Also $T\tilde{\zeta}_\lambda - \frac{1}{2}\lambda r^2 < 0$ on $\Pi\backslash\Pi(\xi)$ hence setting

$$\tilde{\phi}(s) = \begin{cases} \phi(s) & \text{if } s \in \text{dom } \phi, s \geq 0, \\ 0 & \text{if } s < 0, \end{cases}$$

we have $\tilde{\phi}$ is increasing, $\tilde{\zeta}_\lambda = \tilde{\phi} \circ (T\tilde{\zeta}_\lambda - \frac{1}{2}\lambda r^2)$ almost everywhere on Π and

$$\mathcal{L}(T\tilde{\zeta}_\lambda) = \mathcal{L}(K\tilde{\zeta}_\lambda) = \tilde{\zeta}_\lambda = \tilde{\phi} \circ (T\tilde{\zeta}_\lambda - \frac{1}{2}\lambda r^2). \ \Box$$

LEMMA 3.4.16 *Let* $5/2 < p < \infty$. *Let* $\zeta_0 \in L^p(\Pi)$ *be non-negative and vanish outside a set of finite measure* β. *For* $\lambda > 0$ *let* $\xi = \xi(\lambda, \beta)$ *be such that for all* $\zeta \in \overline{R(\zeta_0)}^w$,

$$T\zeta(r, z) - \frac{1}{2}\lambda r^2 < 0 \quad \textit{if } r \geq \xi(\lambda, \beta).$$

Let $\tilde{\zeta}_\lambda$ *be a maximiser of* Ψ_λ *relative to* $\overline{R_\xi(\zeta_0)}^w$. *Then* $\tilde{u} = T\tilde{\zeta}_\lambda$ *satisfies* $\mathcal{L}\tilde{u} = \phi \circ (\tilde{u} - \frac{1}{2}\lambda r^2)$ *almost everywhere in* Π *for some increasing function* ϕ.

Proof By the proof of Theorem 3.4.14, $\tilde{\zeta}_\lambda$ vanishes outside a set of finite measure. Since $\overline{R_\xi(\tilde{\zeta}_\lambda)}^w \subset \overline{R_\xi(\zeta_0)}^w$ and $\overline{R(\tilde{\zeta}_\lambda)}^w \subset \overline{R(\zeta_0)}^w$ we have

$$T\zeta(r, z) - \frac{1}{2}\lambda r^2 < 0 \quad \text{if } r \geq \xi(\lambda, \beta)$$

for all $\zeta \in \overline{R(\tilde{\zeta}_\lambda)}^w$ and $\tilde{\zeta}_\lambda$ maximises Ψ_λ relative to $\overline{R_\xi(\tilde{\zeta}_\lambda)}^w$. Applying Lemma 3.4.15 we obtain the required result. $\Box$

3.4.4 Maximisers for large λ

LEMMA 3.4.17 *Let* $5/2 < p < \infty$. *Let* $\zeta_0 \in L^p(\Pi)$ *be non-negative and vanish outside a set of measure* β. *Then there exists* $\lambda(\beta, ||\zeta_0||_p, p)$ *such that for* $\lambda >$

$\lambda(\beta, ||\zeta_0||_p, p)$, *if* $\tilde{\zeta}_\lambda$ *is a maximiser of* Ψ_λ *relative to* $\overline{R(\zeta_0)}_b^w$, *then* $\tilde{\zeta}_\lambda = 0$.

Proof Let $\xi(\lambda, \beta)$ be such that for all $\zeta \in \overline{R(\zeta_0)}^w$,

$$T\zeta(r,z) - \frac{1}{2}\lambda r^2 < 0 \quad \text{if } r \geq \xi(\lambda, \beta).$$

By Lemma 3.4.8 it is sufficient to show that for functions $v \in \overline{R(\zeta_0)}^w$ supported in the strip $\Pi(\xi(\lambda, \beta))$, $\Psi_\lambda(v) \leq 0$. Also, from Lemma 3.4.7 it follows that there exists $\lambda(\beta, ||\zeta_0||_p, p)$ and a positive constant M such that it is possible to choose $\xi(\lambda, \beta) \leq M$ if $\lambda > \lambda(\beta, ||\zeta_0||_p, p)$. We therefore need only consider functions in $\overline{R(\zeta_0)}^w$ that are supported in $r \leq M$.

Identify $v \in \overline{R(\zeta_0)}^w$ supported in $r \leq M$ with w defined on $\mathbb{R}^5$. Let q denote the conjugate exponent of p. For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^5$ let $\rho = |\mathbf{x} - \mathbf{y}|$. Then

$$\begin{aligned}
8\pi^2 \mathcal{K} w(\mathbf{x}) &= \int_{\mathbb{R}^5} \frac{1}{|\mathbf{x} - \mathbf{y}|^3} w(\mathbf{y}) d\mathbf{y} \\
&= \int_{\rho \geq 1} \frac{1}{|\mathbf{x} - \mathbf{y}|^3} w(\mathbf{y}) d\mathbf{y} + \int_{\rho < 1} \frac{1}{|\mathbf{x} - \mathbf{y}|^3} w(\mathbf{y}) d\mathbf{y} \\
&\leq ||w||_1 + \text{const.} \left(\int_0^1 \rho^{4-3q} d\rho \right)^{1/q} ||w||_p \\
&\leq \text{const.}(||v||_1 + ||v||_p)
\end{aligned}$$

since v is supported in $r \leq M$.

Thus

$$Tv = r^2 \mathcal{K} w \leq \text{const.}(||\zeta_0||_1 + ||\zeta_0||_p) r^2$$

and

$$\begin{aligned}
\Psi_\lambda(v) &= \frac{1}{2} \int_\Pi vTv \; d\nu - \frac{\lambda}{2} \int_\Pi r^2 v \; d\nu \\
&\leq \frac{1}{2} \int_\Pi (\text{const.}(||\zeta_0||_1 + ||\zeta_0||_p) - \lambda) r^2 v \; d\nu \\
&\leq 0
\end{aligned}$$

if λ is sufficiently large as required. □

3.5 The special case when ζ_0 is constant

We show that when ζ_0 is constant in its support and vanishes outside a set of finite measure the maximisers of Ψ_λ relative to $\overline{R(\zeta_0)_b}^w$ are either rearrangements of ζ_0 or identically zero. Furthermore, the values of λ for which there exist maximisers that are rearrangements form an interval.

3.5.1 Hill's spherical vortex and Hill's problem

We state a result of Amick and Fraenkel concerning the relationship between weak solutions of Hill's problem and Hill's spherical vortex.

Let

$$\psi_h(r,z) = \begin{cases} \frac{1}{2}\lambda r^2\left(\frac{5}{2} - \frac{3\rho^2}{2\alpha^2}\right), & \text{if } \rho \le \alpha \\ \frac{1}{2}\lambda r^2 \frac{\alpha^3}{\rho^3} & \text{if } \rho \ge \alpha \end{cases} \tag{3.5.33}$$

denote Hill's vortex where $\rho = (r^2+z^2)^{1/2}$ and let $k = 15\lambda/2\alpha^2$. Then letting f_H denote the Heaviside function, $\psi_h \in C^1(\overline{\Pi}) \cap C^2(\Pi \backslash \partial B_\alpha(0))$ and

$$\begin{aligned} \mathcal{L}\psi_h &= k f_H(\psi_h - \lambda r^2/2) \text{ in } \Pi, \\ \psi_h(0,z) &= 0 \\ \psi_h(r,z) &\to 0 \text{ as } \rho \to \infty. \end{aligned}$$

A non-zero element, ψ, of H is said to be a weak solution of Hill's problem if there exist constants $\kappa \in \mathbb{R}$ and $\lambda > 0$ such that

$$< \psi, h >_H = \kappa \int_{\mathcal{S}(\psi)} h \; d\nu \quad \text{for all } h \in H$$

where $\mathcal{S}(\psi) = \{(r,z) \in \Pi | \psi(r,z) > \lambda r^2/2\}$.

Amick and Fraenkel [3, Theorem 1.1] showed that if ψ is a weak solution of Hill's problem then $\psi(r,z) = \psi_h(r, z - z_0)$ for some $z_0 \in \mathbb{R}$ (and a suitable choice of α).

3.5.2 Maximisers that are rearrangements

It was shown in Lemma 3.4.13 that if a maximiser, $\tilde{\zeta}_\lambda$, of Ψ_λ relative to $\overline{R(\zeta_0)_b}^w$ is not a rearrangement then

$$\tilde{\zeta}_\lambda^{-1}(0,\infty) = (T\tilde{\zeta}_\lambda - \frac{1}{2}\lambda r^2)^{-1}(0,\infty)$$

and $\tilde{\zeta}_\lambda = \phi \circ (T\tilde{\zeta}_\lambda - \frac{1}{2}\lambda r^2)$ almost everywhere in Π for some increasing function ϕ. Therefore, if ζ_0 is constant then ϕ must be f_H and the maximiser gives rise to a weak solution of Hill's problem. Since such a solution must be a translate of Hill's vortex we deduce that the maximiser is supported on a ball centred on the z-axis. An explicit calculation then shows $\Psi_\lambda(\tilde{\zeta}_\lambda) < 0$ and therefore any maximiser which is not a rearrangement is zero.

We perform a preliminary calculation. For $\alpha > 0$ and $z_0 \in \mathbb{R}$ let

$$S_{\alpha,z_0} = \{(r,z) \in \Pi | r^2 + (z - z_0)^2 \leq \alpha^2\}$$

and let B_{α,z_0} denote the ball in $\mathbb{R}^5$ centred at the point on the z-axis with $z = z_0$ and having radius α. If $z_0 = 0$ then this subscript will be omitted.

LEMMA 3.5.1 *Let $k, \alpha > 0$ be constants and let $z_0 \in \mathbb{R}$. Let $v_{\alpha,z_0} = k1_{S_{\alpha,z_0}}$. Then $\Psi_\lambda(v_{\alpha,z_0})$ is independent of z_0,*

$$\Psi_\lambda(v_{\alpha,z_0}) < 0 \text{ if } \lambda > 2k\alpha^2/21$$

and

$$\Psi_\lambda(v_{\alpha,z_0}) > 0 \text{ if } \lambda < 2k\alpha^2/21.$$

Proof Identify v_{α,z_0} with $w = k1_{B_{\alpha,z_0}}$ defined on $\mathbb{R}^5$. Then

$$\begin{aligned}
\Psi_\lambda(v_{\alpha,z_0}) &= \frac{1}{2}\int_\Pi v_{\alpha,z_0} T v_{\alpha,z_0}\, d\nu - \frac{\lambda}{2}\int_\Pi r^2 v_{\alpha,z_0}\, d\nu \\
&= \frac{1}{2}\int_{\mathbb{R}^5} w\mathcal{K}w\, d\mu - \frac{\lambda}{2}\int_\Pi v_{\alpha,z_0} d\tau \\
&= \frac{1}{2\pi}\int_{\mathbb{R}^5} w\mathcal{K}w\, d\mu_5 - \lambda\pi k\int_{S_{\alpha,z_0}} r^3 dr dz
\end{aligned}$$

$$= \frac{k^2}{2\pi}\int_{B_{\alpha,z_0}}\int_{B_{\alpha,z_0}}\frac{1}{8\pi^2|\mathbf{x}-\mathbf{y}|^3}d\mathbf{x}d\mathbf{y} - \lambda\pi k\int_{S_{\alpha,z_0}} r^3 dr dz.$$

Using a change of variables and the formula for the Newtonian potential of a function with constant density on a ball centred at the origin we obtain

$$\begin{aligned}
\Psi_\lambda(v_{\alpha,z_0}) &= \frac{k^2}{2\pi}\int_{B_\alpha}\left(-\frac{|\mathbf{x}|^2}{10}+\frac{\alpha^2}{6}\right)d\mathbf{x} - \lambda\pi k\int_0^\pi\int_0^\alpha \rho^4\sin^3\theta\ d\rho d\theta \\
&= \frac{4\pi k^2}{3}\int_0^\alpha\left(-\frac{u^2}{10}+\frac{\alpha^2}{6}\right)\ u^4 du - \frac{4\lambda k\pi\alpha^5}{15} \\
&= \frac{4k\pi\alpha^5}{15}\left(\frac{2k\alpha^2}{21}-\lambda\right)
\end{aligned}$$

from which we obtain the required result. □

THEOREM 3.5.2 *Let $\zeta_0 = k1_A$ where $\nu(A) = \beta$ is finite. For $\lambda > 0$ let $\tilde{\zeta}_\lambda$ be a maximiser of Ψ_λ relative to $\overline{R(\zeta_0)}_b^w$. Then there exists $\lambda_0 > 0$ such that*

$$\begin{aligned}
\tilde{\zeta}_\lambda \in R(\zeta_0) \quad &\textit{if} \quad 0 < \lambda < \lambda_0, \\
\tilde{\zeta}_\lambda \in R(\zeta_0) \textit{ or } \tilde{\zeta}_\lambda = 0 \quad &\textit{if} \quad \lambda = \lambda_0, \\
\tilde{\zeta}_\lambda = 0 \quad &\textit{if} \quad \lambda > \lambda_0.
\end{aligned}$$

Furthermore

$$\lambda_0 \geq \frac{2k}{21}\left(\frac{3\beta}{4\pi}\right)^{2/3}.$$

Proof By Lemma 3.4.16, $\tilde{\zeta}_\lambda = \phi\circ(T\tilde{\zeta}_\lambda - \frac{1}{2}\lambda r^2)$ almost everywhere on Π.

Suppose for a contradiction that $\tilde{\zeta}_\lambda$ is a rearrangement of a strict non-zero curtailment of ζ_0. Then by Lemma 3.4.13, $\tilde{\zeta}_\lambda^{-1}(0,\infty) = (T\tilde{\zeta}_\lambda - \frac{1}{2}\lambda r^2)^{-1}(0,\infty)$ except for a set of measure zero. From (3.3.11) we have for all $h \in H$

$$\int_\Pi \frac{1}{r^2}\nabla K\tilde{\zeta}_\lambda.\nabla h\ d\nu = \int_\Pi \tilde{\zeta}_\lambda h\ d\nu = k\int_{S(\tilde{\zeta}_\lambda)} h\ d\nu \tag{3.5.34}$$

where $\mathcal{S}(\tilde{\zeta}_\lambda) = \{(r,z)\in\Pi | K\tilde{\zeta}_\lambda(r,z) - \lambda r^2/2 > 0\}$. Hence $K\tilde{\zeta}_\lambda$ is a weak solution of Hill's problem and we deduce $K\tilde{\zeta}_\lambda(r,z) = \psi_h(r, z - z_0)$ for some $z_0 \in \mathbb{R}$ with $\tilde{\zeta}_\lambda = k1_{S_{\alpha,z_0}}$ where $\lambda = 2k\alpha^2/15$.

But by Lemma 3.5.1, $\Psi_\lambda(\tilde{\zeta}_\lambda) < 0$ which contradicts the fact that $\tilde{\zeta}_\lambda$ maximises Ψ_λ relative to $\overline{R(\zeta_0)_b}^w$. Hence $\tilde{\zeta}_\lambda$ can not be a strict non-zero curtailment and is either a rearrangement of ζ_0 or zero. If $\tilde{\zeta}_\lambda = 0$ then $\Psi_\lambda(\zeta) \leq 0$ for all $\zeta \in \overline{R(\zeta_0)_b}^w$.

The calculation performed in Lemma 3.5.1 shows that Ψ_λ is not maximised by zero relative to $\overline{R(\zeta_0)_b}^w$ for sufficiently small λ. Define

$$\lambda_0 = \inf\{\lambda | \sup_{\zeta \in \overline{R(\zeta_0)_b}^w} \Psi_\lambda(\zeta) = 0\}.$$

Then $\lambda_0 > 0$ and for $\lambda > \lambda_1 > \lambda_0$, $\Psi_\lambda(\zeta) < \Psi_{\lambda_1}(\zeta) \leq 0$ for all non-zero $\zeta \in \overline{R(\zeta_0)_b}^w$. Therefore for all $\lambda > \lambda_0$, zero is the unique maximiser of Ψ_λ relative to $\overline{R(\zeta_0)_b}^w$.

We show that when $\lambda = \lambda_0$, Ψ_λ is maximised by zero relative to $\overline{R(\zeta_0)_b}^w$ (but there may be rearrangements that are also maximisers). We recall supp $\tilde{\zeta}_{\lambda_0} \subset \Pi(\xi(\lambda_0))$ for some $\xi(\lambda_0) > 0$. Let $\{\lambda_n\}_{n=1}^\infty$ be a strictly decreasing sequence with $\lambda_n \to \lambda_0$ as $n \to \infty$. Then

$$\begin{aligned}\Psi_{\lambda_0}(\tilde{\zeta}_{\lambda_0}) \leq \Psi_{\lambda_0}(\tilde{\zeta}_{\lambda_0}) - \Psi_{\lambda_n}(\tilde{\zeta}_{\lambda_0}) &= \frac{(\lambda_n - \lambda_0)}{2} \int_\Pi r^2 \tilde{\zeta}_{\lambda_0}\, d\nu \\ &\to 0 \text{ as } n \to \infty.\end{aligned}$$

Let $\beta = \frac{4}{3}\pi a^3$. Then for $\lambda < 2ka^2/21$ we have $\lambda > 2k\alpha^2/21$ for some $\alpha < a$. Let $v_\alpha = k1_{S_{\alpha,z_0}}$. Then $\nu\{v_\alpha > 0\} = \frac{4}{3}\pi\alpha^3$ and therefore $v_\alpha \in RC(\zeta_0)$. Also, by Lemma 3.5.1 $\Psi_\lambda(v_\alpha) > 0$, hence zero does not maximise Ψ_λ relative to $\overline{R(\zeta_0)_b}^w$ for $\lambda < 2ka^2/21$. Thus $\lambda_0 \geq 2ka^2/21$ and substituting $a^2 = (3\beta/4\pi)^{2/3}$ yields the required inequality. □

LEMMA 3.5.3 *Let $5/2 < p < \infty$. Let $\zeta_0 \in L^p(\Pi)$ be non-negative and vanish outside a set of finite measure. Let $\tilde{\zeta}_\lambda$ denote a Steiner-symmetric maximiser of Ψ_λ relative to $\overline{R(\zeta_0)_b}^w$. Then for $\delta > 0$ there exist $\xi(\delta)$, $z(\delta)$ such that for $\lambda > \delta$,*

$$\text{supp } \tilde{\zeta}_\lambda \subset \{(r,z) \in \Pi | 0 < r < \xi(\delta), |z| < z(\delta)\}.$$

except for a set of measure zero.

Proof From Lemma 3.4.7, Lemma 3.4.13 and the fact that

$$\text{supp } \tilde{\zeta}_\lambda \subset \{K\tilde{\zeta}_\lambda - \lambda r^2/2 > 0\} \subset \{K\tilde{\zeta}_\lambda - \delta r^2/2 > 0\} \qquad (3.5.35)$$

where a set of zero measure may be excepted in the first containment, it follows that supp $\tilde{\zeta}_\lambda \subset \Pi(\xi(\delta))$ for some $\xi(\delta) > 0$. Let $\mathbf{x} = (x_1, ..., x_4, z) \in \mathbb{R}^5$ with $z \geq 1$ and define

$$G = \{(y_1, ..., y_4, z') \in \mathbb{R}^5 | |z' - z| < 1\}.$$

Let $5/2 < p < \infty$ and let $w \in L^1(\mathbb{R}^5) \cap L^p(\mathbb{R}^5)$ be non-negative and Steiner-symmetric. Then $\mathcal{K}w$ is non-negative and Steiner-symmetric. Furthermore, for $1 \leq s \leq p$ we have

$$\begin{aligned} ||w||_{s;G} &\leq z^{-1/s}||w||_s \\ ||\mathcal{K}w||_{10/3;G} &\leq z^{-3/10}||\mathcal{K}w||_{10/3}. \end{aligned}$$

We use the methods of Lemma 3.3.8. Letting $B_0 = B_{1/4}(\mathbf{x})$ and $B_3 = B_1(\mathbf{x})$ it follows from (3.3.13) that

$$||\mathcal{K}w||_{2,p;B_0} \leq M(p)(||w||_{p;B_3} + ||\mathcal{K}w||_{10/3;B_3}).$$

Hence

$$\begin{aligned} ||\mathcal{K}w||_{2,p;B_0} &\leq M(p)(||w||_{p;G} + ||\mathcal{K}w||_{10/3;G}) \\ &\leq M(p)(z^{-1/p}||w||_p + z^{-3/10}||\mathcal{K}w||_{10/3}) \\ &\leq M(p)(z^{-1/p}||w||_p + z^{-3/10}||w||_{10/7}) \end{aligned}$$

where in the last inequality we have used the fact that $\mathcal{K} : L^{10/7}(\mathbb{R}^5) \to L^{10/3}(\mathbb{R}^5)$ is bounded.

Since $\tilde{\zeta}_\lambda$ is supported on $\Pi(\xi(\delta))$, identifying $\tilde{\zeta}_\lambda$ with $\tilde{w}_\lambda$ on $\mathbb{R}^5$ and applying the embedding $W^{2,p}(B_0) \to C(\overline{B_0})$, we have for $|z| \geq 1$

$$\mathcal{K}\tilde{w}_\lambda(x_1, ..., x_4, z) \leq M(p, \delta)(|z|^{-1/p}||\zeta_0||_p + |z|^{-3/10}||\zeta_0||_{10/7}).$$

We recall $K\tilde{\zeta}_\lambda = r^2\mathcal{K}\tilde{w}_\lambda$. Hence

$$\begin{aligned} K\tilde{\zeta}_\lambda(r, z) - \frac{1}{2}\delta r^2 &< M(p, \delta)(|z|^{-1/p}||\zeta_0||_p + |z|^{-3/10}||\zeta_0||_{10/7})r^2 - \frac{1}{2}\delta r^2 \\ &< 0 \text{ if } |z| \geq z(\delta) \end{aligned}$$

and the result now follows from (3.5.35).

LEMMA 3.5.4 *Let $\zeta_0 = k1_A$ where $\nu(A)$ is finite. Let λ_0 be as in Theorem 3.5.2. Then there exists $\zeta \in R(\zeta_0)$ such that ζ maximises Ψ_{λ_0} relative to $\overline{R(\zeta_0)}_b^w$.*

Proof Let $0 < \delta < \lambda_0$ and let $\{\lambda_n\}_{n=1}^{\infty}$ be an increasing sequence converging to λ_0 as $n \to \infty$ with $\delta < \lambda_n < \lambda_0$ for all n. Let $\tilde{\zeta}_\lambda$ denote a Steiner-symmetric maximiser of Ψ_λ relative to $\overline{R(\zeta_0)}_b^w$. Then by Lemma 3.5.3 there exist $\xi(\delta)$, $z(\delta)$ such that for $\lambda > \delta$

$$\text{supp } \tilde{\zeta}_\lambda \subset \{(r,z) \in \Pi | 0 < r < \xi(\delta), |z| < z(\delta)\}.$$

By Theorem 3.5.2, $\tilde{\zeta}_{\lambda_n} \in R(\zeta_0)$ for all n hence by weak sequential compactness of $\overline{R(\zeta_0)}^w$, passing to a subsequence if necessary, $\tilde{\zeta}_{\lambda_n} \xrightarrow{w} \zeta$ for some $\zeta \in \overline{R(\zeta_0)}_b^w$ with $||\zeta||_1 = ||\zeta_0||_1$. Then

$$\begin{aligned}
\Psi_{\lambda_0}(\zeta) &= \lim_{n\to\infty} \left\{ \frac{1}{2} \int_\Pi \tilde{\zeta}_{\lambda_n} T \tilde{\zeta}_{\lambda_n} \, d\nu \right\} - \frac{\lambda_0}{2} \int_\Pi r^2 \zeta \, d\nu \\
&\geq \limsup_{n\to\infty} \left\{ \frac{1}{2} \int_\Pi \tilde{\zeta}_{\lambda_n} T \tilde{\zeta}_{\lambda_n} \, d\nu - \frac{\lambda_n}{2} \int_\Pi r^2 \tilde{\zeta}_{\lambda_n} \, d\nu + \frac{(\lambda_n - \lambda_0)}{2} \int_\Pi r^2 \tilde{\zeta}_{\lambda_n} \, d\nu \right\}.
\end{aligned}$$

But

$$\frac{(\lambda_n - \lambda_0)}{2} \int_\Pi r^2 \tilde{\zeta}_{\lambda_n} \, d\nu \to 0 \text{ as } n \to \infty$$

hence

$$\begin{aligned}
\Psi_{\lambda_0}(\zeta) &\geq \limsup_{n\to\infty} \Psi_{\lambda_n}(\tilde{\zeta}_{\lambda_n}) \\
&\geq 0.
\end{aligned}$$

Thus ζ maximises Ψ_{λ_0} relative to $\overline{R(\zeta_0)}_b^w$ and since $||\zeta||_1 = ||\zeta_0||_1$ it follows that $\zeta \in R(\zeta_0)$. $\square$

Chapter 4

Vortex Streets

4.1 Introduction

In this chapter we prove an existence theorem for a steady flow of an ideal fluid in a strip. Let $b, t > 0$ and for $\lambda > 0$ define

$$\Omega^\lambda = \mathbb{R} \times (0, b/\lambda).$$

The stream function $\psi : \Omega^\lambda \to \mathbb{R}$ is such that $\psi(\cdot, x_2)$ is periodic with period t/λ. The vorticity is given by $-\Delta\psi$ and the vorticity restricted to $\Omega_0^\lambda := (-t/2\lambda, t/2\lambda) \times (0, b/\lambda)$ is a rearrangement of a prescribed function. The equation

$$-\Delta\psi = \phi \circ \psi$$

is satisfied almost everywhere in Ω^λ, where ϕ is an increasing function. Furthermore, we prove the existence of flows for which the restriction of the vorticity to Ω_0^λ avoids $\partial\Omega_0^\lambda$ so that the flow contains (disjoint) patches of vorticity, the vorticity in each patch being a rearrangement of a prescribed function.

4.2 Description of the method

Let $w \in L^p(\mathbb{R}^2)$ be non-negative. The Steiner-symmetrisation of w in the line $\{x_1 = 0\}$ is the essentially unique non-negative function $w^\sharp \in L^p(\mathbb{R}^2)$ such that for each $\alpha > 0$ and almost every $x_2 \in \mathbb{R}$ the set

$$\{x_1 | w^\sharp(x_1, x_2) \geq \alpha\}$$

is an interval with centre 0, whose length equals the linear measure of the set

$$\{x_1 | w(x_1, x_2) \geq \alpha\}.$$

Let $2 < p < \infty$. We define an operator $T_\lambda : L^p(\Omega_0^\lambda) \to W^{2,p}(\Omega_0^\lambda)$ such that for $g \in L^p(\Omega_0^\lambda)$ we have

$$\begin{aligned} -\Delta(T_\lambda g) &= g \text{ in } \Omega_0^\lambda \\ \frac{\partial T_\lambda g}{\partial \nu} &= 0 \text{ on } \Gamma_0^\lambda := \partial\Omega_0^\lambda \backslash \partial\Omega^\lambda \\ T_\lambda g &= 0 \text{ on } \partial\Omega_0^\lambda \backslash \Gamma_0^\lambda \end{aligned}$$

where ν denotes the unit exterior normal to $\partial\Omega_0^\lambda$.

Let $u \in W_0^{1,2}(\Omega^\lambda)$ satisfy $-\Delta u = g$ in the weak sense in Ω^λ. For $\mathbf{x} = (x_1, x_2) \in \Omega^\lambda$ and $n \in \mathbb{Z}$ define

$$u_n(x_1, x_2) = u(x_1 - nt/\lambda, x_2)$$

and let

$$v(x_1, x_2) = \sum_{n \in \mathbb{Z}} u_n(x_1, x_2).$$

Then $v(\cdot, x_2)$ is t/λ-periodic, $v \in W_{loc}^{2,p}(\Omega^\lambda)$ and $-\Delta v = g$ almost everywhere in Ω_0^λ. We show that if g is Steiner-symmetric then $T_\lambda g = v|_{\Omega_0^\lambda}$.

Let $\omega_0 \in L^p(\mathbb{R}^2)$ be a non-negative, non-zero function with bounded support. For $g \in L^p(\Omega_0^\lambda)$ define

$$\Psi_\lambda(g) = \frac{1}{2} \int_{\Omega_0^\lambda} g T_\lambda g - \lambda \int_{\Omega_0^\lambda} x_2 g.$$

We show that if the set of rearrangements of ω_0 supported on Ω_0^λ is non-empty then Ψ_λ attains a maximum relative to this set and a maximiser can be chosen to be Steiner-symmetric. Let $\tilde{\omega}$ denote such a maximiser. Support estimates similar to those of Turkington [36, 37] are used to show that the support of $\tilde{\omega}$ is bounded away from $\partial\Omega_0^\lambda$ if λ is sufficiently small and b and t are sufficiently large.

Furthermore, the extension of $T_\lambda\tilde{\omega} - \lambda x_2$ to a function on Ω^λ that is t/λ-periodic in x_1 provides the stream function for a steady flow of an ideal fluid in Ω^λ.

4.3 Existence of steady flows

Let $\Omega = \mathbb{R} \times (0, b) \subset \mathbb{R}^2$. Define

$$\Omega_0 = (-t/2, t/2) \times (0, b)$$

and

$$\Gamma_0 = \partial\Omega_0 \backslash \partial\Omega.$$

Let $C_0^\infty(\Omega_0 \cup \Gamma_0)$ denote those functions $u \in C^\infty(\overline{\Omega_0})$ with supp u a compact subset of $\Omega_0 \cup \Gamma_0$ (equivalently those functions vanishing in a neighbourhood of $\partial\Omega$). In view of the boundary conditions we shall work in the space $W_0^{1,2}(\Omega_0 \cup \Gamma_0)$ which is the closure of $C_0^\infty(\Omega_0 \cup \Gamma_0)$ in $W^{1,2}(\Omega_0)$.

Troianello [35, Lemma 1.46] shows that a Poincaré inequality holds for functions in $W_0^{1,2}(\Omega_0 \cup \Gamma_0)$ and, in particular, a norm on $W_0^{1,2}(\Omega_0 \cup \Gamma_0)$ equivalent to the usual Sobolev norm is given by $||u||_{W_0^{1,2}(\Omega_0 \cup \Gamma_0)} = ||\nabla u||_2$.

For $g \in L^p(\Omega_0)$ let Tg denote the unique minimiser of the functional

$$\psi_g(w) = \frac{1}{2}\int_{\Omega_0} |\nabla w|^2 - \int_{\Omega_0} wg \qquad w \in W_0^{1,2}(\Omega_0 \cup \Gamma_0).$$

LEMMA 4.3.1 *Let $1 < p < \infty$ and let q denote the conjugate exponent of p. Then $T : L^p(\Omega_0) \to L^q(\Omega_0)$ is a compact, symmetric, strictly positive operator. Furthermore*

$$\int_{\Omega_0} \nabla Tg.\nabla h = \int_{\Omega_0} hg \quad \forall h \in W_0^{1,2}(\Omega_0 \cup \Gamma_0).$$

Proof By the embedding $W^{1,2}(\Omega_0) \to L^s(\Omega_0)$, $1 \leq s < \infty$,

$$\begin{aligned}
\psi_g(w) &= \frac{1}{2}||w||^2_{W_0^{1,2}(\Omega_0\cup\Gamma_0)} - \int_{\Omega_0} wg \\
&\geq \frac{1}{2}||w||^2_{W_0^{1,2}(\Omega_0\cup\Gamma_0)} - ||w||_q||g||_p \\
&\geq \frac{1}{2}||w||^2_{W_0^{1,2}(\Omega_0\cup\Gamma_0)} - const.||w||_{W_0^{1,2}(\Omega_0\cup\Gamma_0)}||g||_p
\end{aligned}$$

$$\rightarrow \infty \text{ as } ||w||_{W_0^{1,2}(\Omega_0 \cup \Gamma_0)} \rightarrow \infty.$$

It follows that ψ_g is strictly convex, coercive and weakly lower semicontinuous hence ψ_g has a unique minimiser, Tg, say. Since the Fréchet derivative is zero at the minimiser

$$d\psi_g[Tg](h) = 0 \quad \forall h \in W_0^{1,2}(\Omega_0 \cup \Gamma_0)$$

and

$$\int_{\Omega_0} \nabla Tg.\nabla h = \int_{\Omega_0} hg \quad \forall h \in W_0^{1,2}(\Omega_0 \cup \Gamma_0).$$

Considering $h = 0$ we have

$$||Tg||^2_{W_0^{1,2}(\Omega_0 \cup \Gamma_0)} \leq 2||Tg||_q||g||_p \leq const.||Tg||_{W_0^{1,2}(\Omega_0 \cup \Gamma_0)}||g||_p.$$

Hence $T : L^p(\Omega_0) \rightarrow W_0^{1,2}(\Omega_0 \cup \Gamma_0)$ is bounded and the compactness of the embedding $W_0^{1,2}(\Omega_0 \cup \Gamma_0) \rightarrow L^q(\Omega_0)$ yields that $T : L^p(\Omega_0) \rightarrow L^q(\Omega_0)$ is compact.

For all $g, g_1 \in L^p(\Omega_0)$

$$\int_{\Omega_0} gTg_1 = \int_{\Omega_0} \nabla Tg.\nabla Tg_1 = \int_{\Omega_0} g_1 Tg$$

which shows that T is symmetric. Taking $g = g_1 \neq 0$ we have $Tg \neq 0$ and

$$\int_{\Omega_0} gTg > 0$$

hence T is strictly positive. □

The methods used in Lemma 4.3.2 and Lemma 4.3.3 are similar to those of [10, Lemma 2].

LEMMA 4.3.2 *Let $1 < p < \infty$. Let $g \in L^p(\Omega_0)$ be non-negative and Steiner-symmetric. Then Tg is non-negative and Steiner-symmetric.*

Proof We recall Tg is the unique minimiser over $w \in W_0^{1,2}(\Omega_0 \cup \Gamma_0)$ of the functional

$$\psi_g(w) = \frac{1}{2}\int_{\Omega_0} |\nabla w|^2 - \int_{\Omega_0} wg.$$

From Kawohl [26, Corollary 2.14] it follows that for all $w \in W^{1,2}(\Omega_0)$

$$\int_{\Omega_0} |\nabla w^\sharp|^2 \leq \int_{\Omega_0} |\nabla w|^2. \tag{4.3.1}$$

Let $u = Tg$. Then by the weak maximum principle [35, Theorem 2.3], u is non-negative and

$$\begin{aligned}
\psi_g(u^\sharp) &= \frac{1}{2}\int_{\Omega_0} |\nabla u^\sharp|^2 - \int_{\Omega_0} u^\sharp g \\
&= \frac{1}{2}\int_{\Omega_0} |\nabla u^\sharp|^2 - \int_{\Omega_0} u^\sharp g^\sharp \\
&\leq \frac{1}{2}\int_{\Omega_0} |\nabla u|^2 - \int_{\Omega_0} ug \\
&= \psi_g(u).
\end{aligned}$$

Since the minimiser of ψ_g is unique $u = u^\sharp$. □

LEMMA 4.3.3 *Let $1 < p < \infty$. Let $g \in L^p(\Omega_0)$ be non-negative. Then Tg is non-negative and*

$$\int_{\Omega_0} gTg \leq \int_{\Omega_0} g^\sharp Tg^\sharp.$$

Proof By Lemma 4.3.1,

$$\int_{\Omega_0} gTg = \int_{\Omega_0} |\nabla Tg|^2.$$

Hence

$$-\frac{1}{2}\int_{\Omega_0} gTg = \inf_{w \in W_0^{1,2}(\Omega_0 \cup \Gamma_0)} \frac{1}{2}\int_{\Omega_0} |\nabla w|^2 - \int_{\Omega_0} wg. \tag{4.3.2}$$

Let $u = Tg$. Replace g by $g^\sharp$ in (4.3.2) to obtain

$$\frac{1}{2}\int_{\Omega_0} |\nabla u^\sharp|^2 - \int_{\Omega_0} u^\sharp g^\sharp \geq -\frac{1}{2}\int_{\Omega_0} g^\sharp Tg^\sharp.$$

Also

$$\frac{1}{2}\int_{\Omega_0} |\nabla u|^2 - \int_{\Omega_0} ug = -\frac{1}{2}\int_{\Omega_0} gTg.$$

By the weak maximum principle [35, Theorem 2.3], Tg is non-negative hence we have the inequality

$$\int_{\Omega_0} u^\sharp g^\sharp \geq \int_{\Omega_0} ug.$$

Therefore by (4.3.1)

$$\frac{1}{2}\int_{\Omega_0} g^\sharp Tg^\sharp - \frac{1}{2}\int_{\Omega_0} gTg$$

$$\begin{aligned}
&\geq -\frac{1}{2}\int_{\Omega_0}|\nabla u^\sharp|^2+\int_{\Omega_0}u^\sharp g^\sharp+\frac{1}{2}\int_{\Omega_0}|\nabla u|^2-\int_{\Omega_0}ug \\
&\geq 0. \quad \square
\end{aligned}$$

We shall frequently make use of the following version of the Divergence Theorem (see for example [24, Theorem 1.5.1]).

THEOREM 4.3.4 (The Divergence Theorem) *Let U be an open bounded subset of $\mathbb{R}^n$ with Lipschitz boundary. Let the unit exterior normal at a point $\mathbf{x} \in \partial U$ (which is defined $d\sigma$ almost everywhere on ∂U) be denoted by $\nu : \mathbf{x} \to (\nu_1(\mathbf{x}), ..., \nu_n(\mathbf{x}))$. Let $u, v \in W^{1,2}(U)$. Then*

$$\int_U u_{x_i} v \ d\mathbf{x} = -\int_U u v_{x_i} \ d\mathbf{x} + \int_{\partial U} uv\nu_i \ d\sigma \quad i = 1, ..., n.$$

LEMMA 4.3.5 *Let $\Sigma = (0, c) \times (0, d)$ and $\tilde{\Sigma} = (-c, c) \times (0, d)$. Let $u \in W^{1,2}(\Sigma)$ and let $\tilde{u}$ denote the extension of u, as an even function of x_1, to a function on $\tilde{\Sigma}$. Then $\tilde{u} \in W^{1,2}(\tilde{\Sigma})$, $\tilde{u}_{x_1}$ is an odd function of x_1 and $\tilde{u}_{x_2}$ is an even function of x_1.*

Proof Let $\tilde{u}_{x_1}$ and $\tilde{u}_{x_2}$ denote the odd and even extensions (as functions of x_1) of u_{x_1} and u_{x_2} respectively. We show

$$\int_{\tilde{\Sigma}} \tilde{u}\phi_{x_i} = -\int_{\tilde{\Sigma}} \tilde{u}_{x_i}\phi \quad \forall\phi \in C_0^\infty(\tilde{\Sigma}) \quad i = 1, 2. \tag{4.3.3}$$

By considering the odd and even parts (as functions of x_1) of ϕ it is sufficient to prove (4.3.3) for ϕ odd in the case $i = 1$ and ϕ even in the case $i = 2$. Also we need only consider the integrals over Σ.

Let $\phi \in C_0^\infty(\tilde{\Sigma})$ be an odd function of x_1. Then $\phi \in C^\infty(\overline{\Sigma})$ and $\phi = 0$ on $\partial\Sigma$ hence $\phi \in W_0^{1,2}(\Sigma)$. Let $\{\theta_n\}_{n=1}^\infty$ be a sequence in $C_0^\infty(\Sigma)$ converging to ϕ in $W_0^{1,2}(\Sigma)$. Then

$$\int_\Sigma \tilde{u}(\theta_n)_{x_1} = -\int_\Sigma \tilde{u}_{x_1}\theta_n$$

and letting $n \to \infty$ gives

$$\int_\Sigma \tilde{u}\phi_{x_1} = -\int_\Sigma \tilde{u}_{x_1}\phi$$

which proves (4.3.3) in the case $i = 1$.

For the case $i = 2$ the Divergence Theorem yields

$$0 = \int_{\partial\Sigma} (u\phi)\nu_2 \, d\sigma = \int_{\Sigma} u_{x_2}\phi + u\phi_{x_2} \tag{4.3.4}$$

for all $\phi \in C_0^\infty(\tilde{\Sigma})$. This completes the proof. □

For $\mathbf{x} = (x_1, x_2) \in \Omega_0$ and $n \in \mathbb{Z}$ define

$$\begin{aligned} \tilde{T}g(x_1 + nt, x_2) &= Tg(x_1, x_2) \qquad \text{if } n \text{ is even} \\ \tilde{T}g(x_1 + nt, x_2) &= Tg(-x_1, x_2) \qquad \text{if } n \text{ is odd} \end{aligned} \tag{4.3.5}$$

and, similarly, define

$$\begin{aligned} \tilde{g}(x_1 + nt, x_2) &= g(x_1, x_2) \qquad \text{if } n \text{ is even} \\ \tilde{g}(x_1 + nt, x_2) &= g(-x_1, x_2) \qquad \text{if } n \text{ is odd.} \end{aligned}$$

LEMMA 4.3.6 *Let $1 < p < \infty$ and let $g \in L^p(\Omega_0)$. Then for any bounded open subset U of Ω, $\tilde{T}g \in W^{1,2}(U)$ and*

$$\int_\Omega \nabla\tilde{T}g.\nabla\phi = \int_\Omega \tilde{g}\phi \qquad \forall \phi \in C_0^\infty(\Omega). \tag{4.3.6}$$

Proof That $\tilde{T}g \in W^{1,2}(U)$ follows immediately from Lemma 4.3.5. Let

$$\Sigma = (-t/2, 3t/2) \times (0, b)$$

and

$$\Gamma = \partial\Sigma \backslash \partial\Omega.$$

In order to prove (4.3.6) it is sufficient to prove

$$\int_\Sigma \nabla\tilde{T}g.\nabla\phi = \int_\Sigma \tilde{g}\phi \qquad \forall \phi \in C_0^\infty(\Sigma \cup \Gamma).$$

Let $\phi \in C_0^\infty(\Sigma \cup \Gamma)$ and for $\mathbf{x} = (x_1, x_2) \in \Sigma$ define

$$\phi_{odd}(x_1, x_2) = \frac{1}{2}(\phi(x_1, x_2) - \phi(t - x_1, x_2))$$

$$\phi_e(x_1, x_2) \;=\; \frac{1}{2}(\phi(x_1, x_2) + \phi(t - x_1, x_2))$$

to be the odd and even parts of ϕ respectively. Then

$$\int_\Sigma \nabla\tilde{T}g.\nabla\phi_{odd} = 0.$$

Also

$$\int_\Sigma \nabla\tilde{T}g.\nabla\phi_e = 2\int_{\Omega_0} \nabla\tilde{T}g.\nabla\phi_e.$$

But $\phi_e \in C_0^\infty(\Omega_0 \cup \Gamma_0)$ so that

$$\int_{\Omega_0} \nabla\tilde{T}g.\nabla\phi_e = \int_{\Omega_0} g\phi_e.$$

Therefore

$$\int_\Sigma \nabla\tilde{T}g.\nabla\phi = 2\int_{\Omega_0} g\phi_e = \int_\Sigma \tilde{g}\phi_e = \int_\Sigma \tilde{g}\phi$$

as required. □

LEMMA 4.3.7 *Let $1 < p < \infty$ and let $g \in L^p(\Omega_0)$. Then for any bounded open set $U \subset \Omega$, $\tilde{T}g \in W^{2,p}(U)$, $-\Delta\tilde{T}g = \tilde{g}$ almost everywhere in Ω and*

$$||Tg||_{2,p;\Omega_0} \leq const.(||Tg||_{p;\Omega_0} + ||g||_{p;\Omega_0}).$$

Proof By Lemma 4.3.6, for $\phi \in C^2(\overline{\Omega})$ having bounded support and vanishing on $\partial\Omega$ we have

$$\int_\Omega \tilde{g}\phi = \int_\Omega \nabla\tilde{T}g.\nabla\phi = -\int_\Omega \tilde{T}g(\Delta\phi)$$

where the second equality follows from the Divergence Theorem.

We use a modification of [2, Theorem 8.1]. The interior regularity for Ω_0 follows using the regularity theorems valid in circular domains. The boundary regularity on Ω_0 is established using half-discs, with $W^{2,p}$ bounds in neighbourhoods of the corner points obtained using L^p estimates in half-discs which overlap the neighbouring rectangle. Since the extensions of Tg and g are even we obtain the required estimate. □

4.3.1 Flows with prescribed speed

Let $1 < p < \infty$ and let q denote the conjugate exponent of p. By Lemma 4.3.1 $T : L^p(\Omega_0) \to L^q(\Omega_0)$ is a compact, symmetric, strictly positive operator hence Ψ_λ is a strictly convex weakly sequentially continuous functional on $L^p(\Omega_0)$. By [9, Theorem 7] and Lemma 4.3.7 we have the following result.

THEOREM 4.3.8 *Let $1 < p < \infty$. Let $\omega_0 \in L^p(\Omega_0)$ be non-negative. Let $\mathcal{F}$ be the set of rearrangements of ω_0 on Ω_0. Then Ψ_λ attains a maximum relative to $\mathcal{F}$. If ω is a maximiser, letting $\psi = \tilde{T}\omega$ we have*

$$-\Delta\psi = \phi \circ (\psi - \lambda x_2)$$

almost everywhere in Ω for some increasing function ϕ.

We note that in the above theorem $\psi - \lambda x_2$ represents the stream function for a steady flow of an ideal fluid in Ω. Furthermore, by Lemma 4.3.3 the maximiser, ω, may be chosen to be Steiner-symmetric. Then $T\omega$ is Steiner-symmetric and $\tilde{T}\omega(\cdot, x_2)$ is t-periodic for almost all $x_2 \in (0, b)$. The average velocity over Ω_0 is λ in the negative x_1 direction. We study this problem further in section 4.5.

4.3.2 Flows with prescribed impulse

We briefly mention how a routine application of [9, Theorem 9] yields the existence of flows with vorticity a rearrangement of a prescribed function and prescribed impulse satisfying a feasibilty condition.

In the case of constant vorticity the variational principle is related to that considered by Keady [27] who considers maximisation of a functional relative to a class of sets with fixed centroid and fixed area.

For $g \in L^p(\Omega_0)$ define

$$\Psi(g) = \frac{1}{2}\int_{\Omega_0} gTg$$

and let

$$I(g) = \int_{\Omega_0} x_2 g \qquad g \in L^p(\Omega_0).$$

By Lemma 4.3.1 and Lemma 4.3.7, $T : L^p(\Omega_0) \to L^q(\Omega_0)$ is a compact, positive symmetric operator and $-\Delta\tilde{T}g = \tilde{g}$ almost everywhere in Ω. Applying [9, Theorem 9] immediately yields the following result.

THEOREM 4.3.9 *Let $1 < p < \infty$. Let $\omega_0 \in L^p(\Omega_0)$ be non-negative. Let $\mathcal{F}$ be the set of rearrangements of ω_0 on Ω_0. Let $\alpha \in \mathbb{R}$ and suppose there exist $\omega_1, \omega_2 \in \mathcal{F}$ such that*

$$I(\omega_1) < \alpha < I(\omega_2).$$

Then Ψ attains a maximum relative to the set

$$\{\omega \in \mathcal{F} | I(\omega) = \alpha\}.$$

If ω is a maximiser then $\psi = \tilde{T}\omega$ satisfies

$$-\Delta\psi = \phi \circ (\psi - \lambda x_2)$$

almost everywhere in Ω, for some increasing function ϕ and some real λ.

Since $I(g) = I(g^*)$ for all $g \in L^p(\Omega_0)$, Lemma 4.3.3 ensures that a Steiner-symmetric maximiser exists in Theorem 4.3.9. If ω is a Steiner-symmetric maximiser, then $T\omega$ is also Steiner-symmetric and $\tilde{T}\omega(\cdot, x_2)$ is t-periodic for almost all $x_2 \in (0, b)$.

4.4 A representation for Tg when g is Steiner-symmetric

In this section we show that when g is Steiner-symmetric, $\tilde{T}g$ may be written as a sum of translates of the solution of a Dirichlet problem on Ω.

For $n \in \mathbb{Z}$ define

$$\Omega_n = ((n - 1/2)t, (n + 1/2)t) \times (0, b)$$

and

$$\Gamma_n = \partial\Omega_n \backslash \partial\Omega.$$

Throughout this section $2 < p < \infty$ with q the conjugate exponent of p. Let f be a non-negative function on Ω, t-periodic in x_1 with $f_0 := f|_{\Omega_0} \in L^p(\Omega_0)$ and $\mu_2\{f_0 > 0\} = \pi a^2$. Let $u \in W_0^{1,2}(\Omega)$ satisfy $-\Delta u = f_0$ in the weak sense in Ω.

For $\mathbf{x} = (x_1, x_2) \in \Omega$ and $n \in \mathbb{Z}$ define

$$u_n(x_1, x_2) = u(x_1 - nt, x_2)$$

and let

$$v(x_1, x_2) = \sum_{n \in \mathbb{Z}} u_n(x_1, x_2) = \sum_{n \in \mathbb{Z}} u(x_1 - nt, x_2).$$

Note that v is t-periodic.

For $g \in L^p(\Omega)$ having bounded support define

$$T_0 g(\mathbf{x}) = \frac{1}{2\pi} \int_\Omega \log\left(\frac{|\mathbf{x} - \bar{\mathbf{y}}|}{|\mathbf{x} - \mathbf{y}|}\right) g(\mathbf{y}) d\mathbf{y} = \frac{1}{4\pi} \int_\Omega \log\left(1 + \frac{4x_2 y_2}{|\mathbf{x} - \mathbf{y}|^2}\right) g(\mathbf{y}) d\mathbf{y}.$$

We firstly show that v is well-defined.

LEMMA 4.4.1

$$\begin{aligned} 0 \leq \quad v(\mathbf{x}) \quad &\leq A_1 \frac{b x_2}{t^2} ||f_0||_1 + (A_2 + A_3 \log x_2) ||f_0||_p \quad \textit{if } x_2 \geq a\sqrt{3}, \\ 0 \leq \quad v(\mathbf{x}) \quad &\leq A_1 \frac{b x_2}{t^2} ||f_0||_1 + A_4 x_2^\beta ||f_0||_p \qquad\qquad \textit{if } x_2 \leq a\sqrt{3} \end{aligned}$$

where A_1 is an absolute constant, A_2, A_3 and A_4 are positive constants depending on a and p only, and $0 < \beta < 1$.

Proof Let $f_n = f 1_{\Omega_n}$. Then $u_n \in W_0^{1,2}(\Omega)$ with $-\Delta u_n = f_n$ in Ω in the weak sense. By [11, Lemma 3] $T_0 f_n \in W_{loc}^{2,p}(\Omega)$ with

$$\begin{aligned} T_0 f_n \quad &\geq \quad 0 \quad \text{on } \partial\Omega, \\ -\Delta(T_0 f_n) \quad &= \quad f_n \quad \text{in } \Omega. \end{aligned}$$

By the methods of Douglas [15, Chapter 8, Lemma 6] it follows that for each n

$$0 \leq u_n \leq T_0 f_n \quad \text{almost everywhere in } \Omega.$$

Hence

$$v = \sum_{n \in \mathbb{Z}} u_n \leq \sum_{n \in \mathbb{Z}} T_0 f_n. \tag{4.4.7}$$

Note that

$$\begin{aligned}
T_0 f_0(x_1 - nt, x_2) &= \frac{1}{2\pi}\int_\Omega \log\left(\frac{|(x_1 - nt, x_2) - \bar{\mathbf{y}}|}{|(x_1 - nt, x_2) - \mathbf{y}|}\right) f_0(\mathbf{y})d\mathbf{y} \\
&= \frac{1}{2\pi}\int_\Omega \log\left(\frac{|\mathbf{x} - \bar{\mathbf{y}}|}{|\mathbf{x} - \mathbf{y}|}\right) f_0(y_1 - nt, y_2)d\mathbf{y} \\
&= T_0 f_n(x_1, x_2).
\end{aligned}$$

Let $\mathbf{x} \in \Omega_0$. Then if $|n| > 1$,

$$\begin{aligned}
T_0 f_n(\mathbf{x}) &= \frac{1}{4\pi}\int_{\Omega_n} \log\left(1 + \frac{4x_2y_2}{|\mathbf{x} - \mathbf{y}|^2}\right) f_n(\mathbf{y})d\mathbf{y} \\
&\leq \frac{1}{\pi}\int_{\Omega_n} \frac{x_2y_2}{|\mathbf{x} - \mathbf{y}|^2} f_n(\mathbf{y})d\mathbf{y} \\
&\leq \frac{bx_2}{\pi((|n| - 1)t)^2}||f_0||_1 \qquad (4.4.8)
\end{aligned}$$

since $|\mathbf{x} - \mathbf{y}| > |x_1 - y_1| > (|n| - 1)t$. By (4.4.7)

$$v \leq T_0(f_{-1} + f_0 + f_1) + \sum_{|n|>1} T_0 f_n.$$

The estimate (4.4.8) yields

$$\sum_{|n|>1} T_0 f_n \leq \frac{bx_2}{\pi t^2}||f_0||_1 \sum_{|n|>1} \frac{1}{(|n| - 1)^2}.$$

We observe $(f_{-1} + f_0 + f_1)$ vanishes outside a set of measure $3\pi a^2$ and $||(f_{-1} + f_0 + f_1)||_p = 3^{1/p}||f_0||_p$. Applying [11, Lemma 1]

$$\begin{aligned}
T_0(f_{-1} + f_0 + f_1)(\mathbf{x}) &\leq (A_2 + A_3 \log x_2)||f_0||_p \quad \text{if } x_2 \geq a\sqrt{3}, \\
T_0(f_{-1} + f_0 + f_1)(\mathbf{x}) &\leq A_4 x_2^\beta ||f_0||_p \quad\quad\quad \text{if } x_2 \leq a\sqrt{3}
\end{aligned}$$

where A_2, A_3 and A_4 are positive constants depending on a and p only, and $0 < \beta < 1$.

Combining these estimates gives the required bound for $\mathbf{x} \in \Omega_0$ and by periodicity the estimate holds for all $\mathbf{x} \in \Omega$. □

We will show that for any bounded open subset U of Ω, $v \in W^{2,p}(U)$ and

$-\Delta v = f$ almost everywhere. We require asymptotic estimates for u and $|\nabla u|$.

LEMMA 4.4.2 *If* $|x_1| > 3t/2$, *then*

$$u(\mathbf{x}) \leq \frac{4b^2}{\pi |x_1|^2} ||f_0||_1.$$

Proof If $|x_1| > 3t/2$ and $\mathbf{y} \in \Omega_0$, then $|\mathbf{x} - \mathbf{y}| > |x_1|/2$ and

$$\begin{aligned} u(\mathbf{x}) &\leq \frac{1}{\pi} \int_{\Omega_0} \frac{x_2 y_2}{|\mathbf{x} - \mathbf{y}|^2} f_0(\mathbf{y}) d\mathbf{y} \\ &\leq \frac{4b^2}{\pi |x_1|^2} ||f_0||_1. \ \square \end{aligned}$$

For $\xi > 0$ let

$$\Omega(\xi) = \{\mathbf{x} \in \Omega | |x_1| < \xi\}$$

and

$$\Gamma(\xi) = \partial\Omega(\xi) \backslash \partial\Omega.$$

LEMMA 4.4.3 $u \in W^{2,p}(\Omega)$ *with*

$$||u||_{2,p:\Omega} \leq M(b,p)(||f_0||_2 + ||f_0||_p). \tag{4.4.9}$$

Also

$$|\nabla u(\mathbf{x})| \leq M(b,t,p) \frac{||f_0||_1}{|x_1|^2} \qquad \text{if } |x_1| > 5t/2.$$

Proof The proof of (4.4.9) is similar to that of [10, Lemma 3]. The function u is characterised as the unique minimiser over $w \in W_0^{1,2}(\Omega)$ of the functional

$$P(w) = \frac{1}{2} \int_\Omega |\nabla w|^2 - \int_\Omega w f_0.$$

By Poincaré's inequality and Hölder's inequality

$$||u||^2_{W^{1,2}(\Omega)} = \int_\Omega |u|^2 + |\nabla u|^2$$

$$\begin{aligned} &\leq (b^2+1)\int_\Omega |\nabla u|^2 \\ &\leq 2(b^2+1)||u||_2||f_0||_2. \end{aligned}$$

By [1, Lemma 5.14] the embedding constant associated with the embedding $W^{1,2}(\Omega) \to L^s(\Omega)$, $2 \leq s < \infty$, depends only on s and the cone used to determine the cone property for Ω. Hence

$$||u||_{W^{1,2}(\Omega)} \leq M(b)||f_0||_2$$

and

$$||u||_p \leq M(b,p)||f_0||_2. \tag{4.4.10}$$

Since u minimises P,

$$\int_\Omega \nabla u.\nabla h = \int_\Omega hf_0 \quad \forall h \in W_0^{1,2}(\Omega).$$

Let $\phi \in C^2(\overline{\Omega})$ have bounded support and vanish on $\partial\Omega$. Then by the Divergence Theorem

$$-\int_\Omega u\Delta\phi = \int_\Omega \nabla u.\nabla\phi = \int_\Omega \phi f_0.$$

For $0 < \xi < \xi_1$ a modification of Agmon [2, Theorem 8.1] (similar to that in Lemma 4.3.7) may be used to show $u \in W^{2,p}(\Omega(\xi))$ with

$$||u||_{2,p;\Omega(\xi)} \leq M(\xi,\xi_1,b,p)(||u||_{p;\Omega(\xi_1)} + ||f_0||_{p;\Omega(\xi_1)}).$$

Choose $\xi_1 < 2\xi$ and cover Ω with translates $\Omega^k(\xi)$ of $\Omega(\xi)$ where for $\tau > 0$ and $k \in \mathbb{Z}$

$$\Omega^k(\tau) = \{\mathbf{x} \in \Omega | (2k-1)\tau < x_1 < (2k+1)\tau\}.$$

Then

$$\sum_{k\in\mathbb{Z}}\sum_{|\alpha|\leq 2}\int_{\Omega^k(\xi)} |D^\alpha u|^p \leq M(\xi,\xi_1,b,p)\sum_{k\in\mathbb{Z}}\left(\int_{|x_1-2k\xi|<\xi_1} |u|^p + |f_0|^p\right)$$

hence

$$\sum_{|\alpha|\leq 2}\int_\Omega |D^\alpha u|^p \leq M(\xi,\xi_1,b,p)\left(\int_\Omega |u|^p + |f_0|^p\right).$$

Therefore

$$||u||_{2,p;\Omega} \le M(b,p)(||u||_{p;\Omega} + ||f_0||_{p;\Omega}).$$

The estimate (4.4.9) now follows from (4.4.10).

Let

$$U_n = \Omega_{n-1} \cup \Omega_n \cup \Omega_{n+1}.$$

Let $|n| \ge 3$. Then the same method as that used above shows

$$||u||_{2,p;\Omega_n} \le M(b,t,p)||u||_{p;U_n}$$

since f_0 vanishes outside Ω_0. Also, for $\mathbf{x} \in U_n$,

$$\frac{|n|t}{2} < |x_1| < \frac{3|n|t}{2} \tag{4.4.11}$$

and the estimate in Lemma 4.4.2 yields

$$u(\mathbf{x}) \le \frac{16b^2}{\pi n^2 t^2}||f_0||_1.$$

Hence

$$||u||_{2,p;\Omega_n} \le M(b,t,p)\frac{||f_0||_1}{n^2t^2}.$$

The gradient bound follows from (4.4.11) and the embedding $W^{2,p}(\Omega_n) \to C^{1,\alpha}(\overline{\Omega_n})$, $0 < \alpha \le 1 - 2/p$. □

LEMMA 4.4.4 *For each $\xi > 0$, $v \in W^{2,p}(\Omega(\xi)) \cap W_0^{1,2}(\Omega(\xi) \cup \Gamma(\xi))$ and*

$$||v||_{2,p;\Omega_0} \le M(b,t,p)||f_0||_{p;\Omega_0}. \tag{4.4.12}$$

Moreover, $-\Delta v = f$ almost everywhere in Ω.

Proof Let

$$w_m = \sum_{i=-m}^{m} u_i.$$

For $n > m$,

$$||w_n - w_m||_{2,p;\Omega(\xi)} = ||\sum_{m<|i|\le n} u_i||_{2,p;\Omega(\xi)} \le \sum_{m<|i|\le n} ||u_i||_{2,p;\Omega(\xi)}.$$

From the methods used in Lemma 4.4.3

$$||u_i||_{2,p;\Omega(\xi)} \leq M(\xi, b, p)(||u_i||_{p;\Omega(2\xi)} + ||f_i||_{p;\Omega(2\xi)})$$

where $f_i = f 1_{\Omega_i}$. By Lemma 4.4.2 if $|i| > M(t, \xi)$

$$||u_i||_{p;\Omega(2\xi)} \leq M(\xi, b, p, f_0)\frac{1}{i^2 t^2}.$$

Hence for m sufficiently large, $n > m$,

$$||w_n - w_m||_{2,p;\Omega(\xi)} \leq M(\xi, b, p, f_0) \sum_{m<|i|\leq n} \frac{1}{i^2 t^2}$$

and w_m is a Cauchy sequence in $W^{2,p}(\Omega(\xi))$. Thus $w_m \to w$ in $W^{2,p}(\Omega(\xi))$ for some function w and, in particular, w_m converges pointwise to w in $\Omega(\xi)$. But w_m converges to v pointwise in $\Omega(\xi)$ hence $w_m \to v$ in $W^{2,p}(\Omega(\xi))$.

It can similarly be shown that $v \in W_0^{1,2}(\Omega(\xi) \cup \Gamma(\xi))$.

Note that

$$||w_m||_{2,p;\Omega_0} \leq \sum_{i=-m}^{m} ||u||_{2,p;\Omega_i}. \tag{4.4.13}$$

Letting $U_i = \Omega_{i-1} \cup \Omega_i \cup \Omega_{i+1}$

$$||u||_{2,p;\Omega_i} \leq M(b, t, p)(||u||_{p;U_i} + ||f_0||_{p;U_i}).$$

If $|i| \geq 4$, for all $\mathbf{x} \in U_i$ we have $|x_1| > 5t/2$ and $|x_1| > |i|t/2$ so that the asymptotic estimate of Lemma 4.4.2 yields

$$||u||_{p;U_i} \leq M(b, t, p)\frac{||f_0||_1}{i^2 t^2}.$$

Thus

$$||u||_{2,p;\Omega_i} \leq M(b, t, p)\frac{||f_0||_1}{i^2 t^2} \quad \text{if } |i| \geq 4.$$

Letting $m \to \infty$ in (4.4.13)

$$||v||_{2,p;\Omega_0} \leq M(b, t, p)||f_0||_1 + \sum_{i=-3}^{3} ||u||_{2,p;\Omega_i}.$$

Applying the estimate in Lemma 4.4.3 and Hölder's inequality we obtain (4.4.12).

Finally, let $\phi \in C_0^\infty(\Omega)$. Then

$$\int_\Omega v(-\Delta\phi) = \int_\Omega (\sum_{n\in\mathbb{Z}} u_n)(-\Delta\phi) = \sum_{n\in\mathbb{Z}} \int_\Omega u_n(-\Delta\phi) = \sum_{n\in\mathbb{Z}} \int_\Omega f_n\phi = \int_\Omega f\phi.$$

Therefore $-\Delta v = f$ almost everywhere in Ω. □

LEMMA 4.4.5 *If f_0 is Steiner-symmetric then $v(\cdot, x_2)$ is even for almost all $x_2 \in (0, b)$ and*

$$\int_{\Omega_0} \nabla v.\nabla\phi = \int_{\Omega_0} \phi f_0 \qquad \forall\phi \in C_0^\infty(\Omega_0 \cup \Gamma_0).$$

Proof By the methods of Lemma 4.3.2 it can be shown that if f_0 is Steiner-symmetric then u is Steiner-symmetric. It follows that $v(\cdot, x_2)$ is even for almost all $x_2 \in (0, b)$.

Let $\phi \in C_0^\infty(\Omega_0 \cup \Gamma_0)$ and let ϕ_{odd} and ϕ_e denote respectively the odd and even parts (as functions of x_1) of ϕ. Then by the Divergence Theorem, letting ν denote the unit exterior normal to Γ_0,

$$\begin{aligned}
\int_{\Omega_0} \nabla v.\nabla\phi &= \int_{\Omega_0} \nabla v.\nabla\phi_e \\
&= \int_{\Omega_0} \sum_{n\in\mathbb{Z}} \nabla u_n.\nabla\phi_e \\
&= \sum_{n\in\mathbb{Z}} -\int_{\Omega_0} \phi_e \Delta u_n + \int_{\Gamma_0} \phi_e \frac{\partial u_n}{\partial\nu} \\
&= \sum_{n\in\mathbb{Z}} \int_{\Omega_0} \phi_e f_n + \int_{\Gamma_0} \phi_e \frac{\partial u_n}{\partial\nu} \\
&= \int_{\Omega_0} \phi_e f_0 + \sum_{n\in\mathbb{Z}} \int_{\Gamma_0} \phi_e \frac{\partial u_n}{\partial\nu}.
\end{aligned}$$

Since

$$\frac{\partial u_{n+1}}{\partial x_1}(t/2, x_2) = \frac{\partial u_n}{\partial x_1}(-t/2, x_2)$$

and ϕ_e is even we have

$$\begin{aligned}
\left|\int_{\Gamma_0}\sum_{n=-k}^{k}\phi_e\frac{\partial u_n}{\partial\nu}\right| &= \left|\int_{x_1=t/2}\phi_e\sum_{n=-k}^{k}\frac{\partial u_n}{\partial x_1}-\int_{x_1=-t/2}\phi_e\sum_{n=-k}^{k}\frac{\partial u_n}{\partial x_1}\right| \\
&= \left|\int_{x_1=t/2}\phi_e\frac{\partial u_{-k}}{\partial x_1}-\int_{x_1=-t/2}\phi_e\frac{\partial u_k}{\partial x_1}\right| \\
&\to 0 \quad \text{as } k\to\infty
\end{aligned}$$

hence

$$\int_{\Omega_0}\nabla v.\nabla\phi=\int_{\Omega_0}\phi_e f_0=\int_{\Omega_0}\phi f_0.\ \square$$

LEMMA 4.4.6 *If f_0 is Steiner-symmetric then*

$$Tf_0=v=\sum_{n\in\mathbb{Z}}u_n$$

hence v is Steiner-symmetric.

Proof By Lemma 4.4.5

$$\int_{\Omega_0}\nabla v.\nabla h=\int_{\Omega_0}hf_0 \qquad \forall h\in W_0^{1,2}(\Omega_0\cup\Gamma_0).$$

Recall ψ_{f_0} defined by

$$\psi_{f_0}(w)=\frac{1}{2}\int_{\Omega_0}|\nabla w|^2-\int_{\Omega_0}wf_0$$

is a convex, Gateaux-differentiable functional on $W_0^{1,2}(\Omega_0\cup\Gamma_0)$. Since $d\psi_{f_0}[v](h)=0$ for all $h\in W_0^{1,2}(\Omega_0\cup\Gamma_0)$ we deduce $v=Tf_0$. $\square$

Note that when f_0 is Steiner-symmetric the periodicity of v ensures that $v=\tilde{T}f_0$ where $\tilde{T}f_0$ is the extension of Tf_0 as defined in (4.3.5).

4.5 Vortex patches

Let $2<p<\infty$ and let q denote the conjugate exponent of p. Let $\omega_0\in L^p(\mathbb{R}^2)$ be non-negative with $\mu_2\{\omega_0>0\}=\pi a^2$. Let $b,t>\max\{4,2a\}$ and for $\lambda>0$ define

$$\Omega_0^\lambda=(-t/2\lambda,t/2\lambda)\times(0,b/\lambda)$$

and

$$\Gamma_0^\lambda = \{\mathbf{x} \in \partial\Omega_0^\lambda | 0 < x_2 < b/\lambda\}.$$

Let $T : L^p(\Omega_0) \to W^{2,p}(\Omega_0)$ be defined as in Section 4.3 and recall $T_\lambda : L^p(\Omega_0^\lambda) \to W^{2,p}(\Omega_0^\lambda)$ satisfies

$$\begin{aligned} -\Delta(T_\lambda g) &= g \text{ in } \Omega_0^\lambda \\ \frac{\partial T_\lambda g}{\partial \nu} &= 0 \text{ on } \Gamma_0^\lambda \\ T_\lambda g &= 0 \text{ on } \partial\Omega_0^\lambda \backslash \Gamma_0^\lambda \end{aligned}$$

where ν denotes the unit exterior normal to $\partial\Omega_0^\lambda$. Also, for $g \in L^p(\Omega_0^\lambda)$

$$\Psi_\lambda(g) = \frac{1}{2}\int_{\Omega_0^\lambda} gT_\lambda g - \lambda \int_{\Omega_0^\lambda} x_2 g.$$

For $0 < \lambda < 1$ let $c = 1/\lambda$. For $g \in L^p(\Omega_0^\lambda)$ define $g_c(\mathbf{x}) = c^2 g(c\mathbf{x})$ and let $w(\mathbf{x}) = (T_\lambda g)(c\mathbf{x})$. Then $||g_c||_1 = ||g||_1$, $||g_c||_p = c^{2/q}||g||_p$ and

$$\begin{aligned} -\Delta w &= g_c \text{ in } \Omega_0 \\ \frac{\partial w}{\partial \nu} &= 0 \text{ on } \Gamma_0 \text{ (where } \nu \text{ is the unit exterior normal to } \partial\Omega_0) \\ w &= 0 \text{ on } \partial\Omega_0 \backslash \Gamma_0 \end{aligned}$$

hence $w = Tg_c$.

For $f \in L^p(\Omega_0)$ define

$$\tilde{\Psi}_\lambda(f) = \frac{1}{2}\int_{\Omega_0} fTf - \int_{\Omega_0} x_2 f.$$

Then

$$\Psi_\lambda(g) = \tilde{\Psi}_\lambda(g_c) \text{ for all } g \in L^p(\Omega_0^\lambda).$$

Let $\zeta_c(\mathbf{x}) = c^2\omega_0(c\mathbf{x})$. Then $||\zeta_c||_1 = ||\omega_0||_1$, $||\zeta_c||_p = c^{2/q}||\omega_0||_p$ and $c^2\mu_2\{\zeta_c > 0\} = \mu_2\{\omega_0 > 0\}$. Let $\mathcal{F}$ denote the set of rearrangements of ω_0 supported on Ω_0^λ and let $\mathcal{F}_c$ denote the set of rearrangements of ζ_c supported on Ω_0 (both $\mathcal{F}$ and $\mathcal{F}_c$ are non-empty by the lower bound on b and t).

By Lemma 4.3.1 $T_\lambda : L^p(\Omega_0^\lambda) \to L^q(\Omega_0^\lambda)$ is a compact, symmetric, strictly positive operator hence Ψ_λ is a strictly convex weakly sequentially continuous functional on $L^p(\Omega_0^\lambda)$. By [9, Theorem 7] Ψ_λ attains a maximum relative to $\mathcal{F}$ and, in particular, Lemma 4.3.3 ensures the existence of a Steiner-symmetric maximiser $\tilde{\omega}$ with

$$\tilde{\omega} = \phi \circ (T_\lambda \tilde{\omega} - \lambda x_2)$$

almost everywhere in Ω_0^λ for some increasing function ϕ. Let $\tilde{\zeta}_c(\mathbf{x}) = c^2\tilde{\omega}(c\mathbf{x})$. Then $\tilde{\zeta}_c$ maximises $\tilde{\Psi}_\lambda$ relative to $\mathcal{F}_c$ and, conversely, if $\tilde{\zeta}$ maximises $\tilde{\Psi}_\lambda$ relative to $\mathcal{F}_c$ then

$$\omega(\mathbf{x}) = \frac{1}{c^2}\tilde{\zeta}(\frac{\mathbf{x}}{c})$$

maximises Ψ_λ relative to $\mathcal{F}$.

As observed in 4.3.1 the extension of $T_\lambda\tilde{\omega} - \lambda x_2$ to a function on Ω^λ that is t/λ-periodic in x_1 provides the stream function for a steady flow in Ω^λ. We show that for c sufficiently large the support of any Steiner-symmetric maximiser of $\tilde{\Psi}_\lambda$ relative to $\mathcal{F}_c$ is bounded away from $\partial\Omega_0$ hence there exist flows in Ω_0^λ containing patches of vorticity, the vorticity in each patch being a rearrangement of ω_0.

LEMMA 4.5.1 *For $c > 3a$ we have*

$$\tilde{\Psi}_\lambda(\tilde{\zeta}_c) \geq \frac{1}{4\pi}\log\left(\frac{c}{48a}\right)||\tilde{\zeta}_c||_1^2 - \frac{4}{3}||\tilde{\zeta}_c||_1.$$

Proof Let $U = \Omega_0 \cap B_2(0)$ where $B_2(0)$ is the ball of radius 2 centred at the origin. Let $G_{B_2}(\mathbf{x},\mathbf{y})$ and $G_U(\mathbf{x},\mathbf{y})$ denote the Green's functions for $-\Delta$ with zero Dirichlet boundary conditions on $B_2(0)$ and U respectively. Then

$$G_{B_2}(\mathbf{x},\mathbf{y}) = \frac{1}{2\pi}\log\frac{|2^{-1}|\mathbf{y}|\mathbf{x} - 2|\mathbf{y}|^{-1}\mathbf{y}|}{|\mathbf{x}-\mathbf{y}|}$$

and

$$\begin{aligned} G_U(\mathbf{x},\mathbf{y}) &= G_{B_2}(\mathbf{x},\mathbf{y}) - G_{B_2}(\mathbf{x},\bar{\mathbf{y}}) \\ &= \frac{1}{2\pi}\log\frac{|\mathbf{x}-\bar{\mathbf{y}}|}{|\mathbf{x}-\mathbf{y}|} - \frac{1}{2\pi}\log\frac{||\mathbf{y}|^2\mathbf{x} - 4\bar{\mathbf{y}}|}{||\mathbf{y}|^2\mathbf{x} - 4\mathbf{y}|}. \end{aligned}$$

Let $\hat{\zeta}_c$ denote the circularly symmetric decreasing rearrangement of $\tilde{\zeta}_c$ relative to the point $\mathbf{z} = (0,1)$. Since $T\hat{\zeta}_c$ is non-negative in U the maximum principle yields

$$
\begin{aligned}
\tilde{\Psi}_\lambda(\tilde{\zeta}_c) &\geq \tilde{\Psi}_\lambda(\hat{\zeta}_c) \\
&= \frac{1}{2}\int_{\Omega_0} \hat{\zeta}_c T\hat{\zeta}_c - \int_{\Omega_0} x_2\hat{\zeta}_c \\
&\geq \frac{1}{2}\int_U G_U(\mathbf{x},\mathbf{y})\hat{\zeta}_c(\mathbf{x})\hat{\zeta}_c(\mathbf{y})d\mathbf{x}d\mathbf{y} - \int_U x_2\hat{\zeta}_c \\
&= \frac{1}{4\pi}\int_U \left(\log\left(\frac{|\mathbf{x}-\bar{\mathbf{y}}|}{|\mathbf{x}-\mathbf{y}|}\right) - \log\left(\frac{||\mathbf{y}|^2\mathbf{x}-4\bar{\mathbf{y}}|}{||\mathbf{y}|^2\mathbf{x}-4\mathbf{y}|}\right)\right)\hat{\zeta}_c(\mathbf{x})\hat{\zeta}_c(\mathbf{y})d\mathbf{x}d\mathbf{y} \\
&\quad -\int_U x_2\hat{\zeta}_c. \qquad (4.5.14)
\end{aligned}
$$

For all $\mathbf{x},\mathbf{y} \in \operatorname{supp}\,\hat{\zeta}_c$ the choice of c implies $|\mathbf{x}-\mathbf{y}| < 2a/c$, $|\mathbf{x}-\bar{\mathbf{y}}| \geq 4/3$ and

$$
\begin{aligned}
||\mathbf{y}|^2\mathbf{x}-4\bar{\mathbf{y}}| &\leq 8, \\
|4\mathbf{y}-|\mathbf{y}|^2\mathbf{x}| &\geq 4|\mathbf{y}|-|\mathbf{y}|^2|\mathbf{x}| \\
&\geq \frac{8}{3}-\left(\frac{4}{3}\right)^3 \\
&> \frac{1}{4}.
\end{aligned}
$$

Applying these estimates to (4.5.14) and observing $x_2 < 4/3$ for all $\mathbf{x} \in \operatorname{supp}\,\hat{\zeta}_c$ gives the required result. □

Let $S_c = \{\tilde{\zeta}_c > 0\}$. We recall the "essential diameter" of S_c is defined as

$$\text{ess diam}(S_c) = \min\{d | S_c = S_c' \cup N \text{ with } \operatorname{diam}(S_c') = d \text{ and } \mu_2(N) = 0\}$$

where for $A \subset \mathbb{R}^2$

$$\operatorname{diam}(A) = \sup\{|\mathbf{x}-\mathbf{y}|\,|\,\mathbf{x},\mathbf{y} \in A\}.$$

LEMMA 4.5.2 *Suppose $b/t \leq \alpha_1$. Then there exist constants $\sigma, C_0 > 0$ depending on ω_0, p and α_1 only such that for $c > C_0$*

$$\textit{ess diam}(S_c) \leq \frac{\sigma a t}{c}.$$

Proof By [9, Theorem 7],

$$\tilde{\zeta}_c = \phi \circ (T\tilde{\zeta}_c - x_2) \tag{4.5.15}$$

almost everywhere in Ω_0 for some increasing function ϕ. Hence there exists $\gamma \in \mathbb{R}$ such that

$$S_c = \{\mathbf{x} \in \Omega_0 | T\tilde{\zeta}_c(\mathbf{x}) - x_2 > \gamma\} \tag{4.5.16}$$

except for a set of zero measure. To see this let $L = \{\mathbf{x} \in \Omega_0 | T\tilde{\zeta}_c(\mathbf{x}) - x_2 = \gamma\}$. Then by [23, Lemma 7.7], $\tilde{\zeta}_c(\mathbf{x}) = -\Delta T\tilde{\zeta}_c(\mathbf{x}) = 0$ for almost all $\mathbf{x} \in L$.

If $\gamma < 0$ then

$$0 < x_2 < |\gamma| \Rightarrow T\tilde{\zeta}_c(\mathbf{x}) - x_2 > T\tilde{\zeta}_c(\mathbf{x}) - |\gamma| > \gamma.$$

By considering the area of S_c it follows that $t|\gamma| \leq \pi a^2/c^2$. Hence $|\gamma| < 1/2$ if $c \geq C_1 = (\pi a)^{1/2}$.

Now consider

$$\begin{aligned} F(\tilde{\zeta}_c) &:= \frac{1}{2}\int_{\Omega_0} (T\tilde{\zeta}_c - x_2 - \gamma)\tilde{\zeta}_c \\ &\leq \frac{1}{2}\int_{\Omega_0} (T\tilde{\zeta}_c - x_2 - \gamma - 1)^+\tilde{\zeta}_c + \frac{1}{2}\int_{\Omega_0} \tilde{\zeta}_c. \end{aligned} \tag{4.5.17}$$

Let $\psi = T\tilde{\zeta}_c - x_2 - \gamma - 1$. Then $\psi^+ = 0$ on $\partial\Omega_0 \backslash \Gamma_0$ and by the Divergence Theorem

$$\begin{aligned} \int_{\Omega_0} |\nabla\psi^+|^2 &= \int_{\Omega_0} \nabla\psi^+ . \nabla\psi \\ &= -\int_{\Omega_0} \psi^+\Delta\psi + \int_{\partial\Omega_0} \psi^+\nabla\psi . \nu \\ &= \int_{\Omega_0} \psi^+\tilde{\zeta}_c \\ &\leq \left(\int_{\Omega_0} |\psi^+|^2\right)^{1/2} ||\tilde{\zeta}_c||_2 \end{aligned} \tag{4.5.18}$$

By [1, Lemma 5.14] $W^{1,1}(\Omega_0) \to L^2(\Omega_0)$ and the embedding constant depends only on the cone determining the cone property for Ω_0. Since $b, t > 4$

$$\left(\int_{\Omega_0} |\psi^+|^2\right)^{1/2} \leq k\left(\int_{\Omega_0} |\psi^+| + |\psi^+_{x_1}| + |\psi^+_{x_2}|\right) \tag{4.5.19}$$

for some embedding constant k.

An application of Hölder's inequality yields

$$\left(\int_{\Omega_0} |\psi^+|^2\right)^{1/2} \leq k\left(\frac{\pi a^2}{c^2}\right)^{1/2}\left(\int_{\Omega_0} |\psi^+|^2\right)^{1/2} + k\left(\int_{\Omega_0} \left|\psi^+_{x_1}\right| + \left|\psi^+_{x_2}\right|\right).$$

If $c > 2ka\sqrt{\pi}$ then

$$\begin{aligned}\left(\int_{\Omega_0} |\psi^+|^2\right)^{1/2} &\leq 2k\left(\int_{\Omega_0} \left|\psi^+_{x_1}\right| + \left|\psi^+_{x_2}\right|\right) \\ &\leq 4k\left(\frac{\pi a^2}{c^2}\right)^{1/2}\left(\int_{\Omega_0} \left|\psi^+_{x_1}\right|^2 + \left|\psi^+_{x_2}\right|^2\right)^{1/2} \qquad (4.5.20)\end{aligned}$$

since $\{\mathbf{x} \in \Omega_0 | |\nabla\psi^+(\mathbf{x})| > 0\} \subset S_c$ except for a set of zero measure.

Combining (4.5.18) and (4.5.20)

$$\int_{\Omega_0} |\nabla\psi^+|^2 \leq 4ka\sqrt{\pi}||\omega_0||_2\left(\int_{\Omega_0} |\nabla\psi^+|^2\right)^{1/2}$$

and therefore

$$\int_{\Omega_0} |\nabla\psi^+|^2 \leq \beta$$

where $\beta = 16k^2\pi a^2||\omega_0||_2^2$. In particular,

$$\int_{\Omega_0} \psi^+\tilde{\zeta}_c \leq 4ka\sqrt{\pi}||\omega_0||_2\left(\int_{\Omega_0} |\nabla\psi^+|^2\right)^{1/2} \leq \beta,$$

for all $c \geq C_2 = \max\{C_1, 2ka\sqrt{\pi}\}$. From (4.5.17) we have

$$F(\tilde{\zeta}_c) \leq \frac{1}{2}(\beta + ||\omega_0||_1). \qquad (4.5.21)$$

We observe that

$$\begin{aligned}\gamma||\tilde{\zeta}_c||_1 &= 2\tilde{\Psi}_\lambda(\tilde{\zeta}_c) - 2F(\tilde{\zeta}_c) + \int_{\Omega_0} x_2\tilde{\zeta}_c \qquad (4.5.22) \\ &\geq 2\tilde{\Psi}_\lambda(\tilde{\zeta}_c) - 2F(\tilde{\zeta}_c).\end{aligned}$$

We denote by M any constant independent of c but possibly depending on a, p,

$||\omega_0||_1$, $||\omega_0||_p$ and α_1. If $c > C_3 = \max\{C_2, 3a\}$ then by Lemma 4.5.1 and (4.5.21)

$$\begin{aligned}\gamma &\geq \frac{1}{2\pi}\log\left(\frac{c}{48a}\right)||\tilde{\zeta}_c||_1 - \frac{\beta}{||\omega_0||_1} - \frac{11}{3} \\ &= \frac{1}{2\pi}\log\left(\frac{c}{2a}\right)||\tilde{\zeta}_c||_1 - M. \qquad (4.5.23)\end{aligned}$$

Let $R \geq 1$. For $\mathbf{x} \in \Omega_0$, let $B = \{\mathbf{y} \in \Omega_0 | |\mathbf{y} - \mathbf{x}| < Ra/c\}$. Then by (4.5.16) and (4.5.23)

$$T\tilde{\zeta}_c(\mathbf{x}) - \frac{1}{2\pi}\log\left(\frac{c}{2a}\right)||\tilde{\zeta}_c||_1 > x_2 - M \qquad (4.5.24)$$

for almost all $\mathbf{x} \in S_c$. For $n \in \mathbb{Z}$ define

$$(\tilde{\zeta}_c)_n(x_1, x_2) = \tilde{\zeta}_c(x_1 - nt, x_2)$$

and note that $(\tilde{\zeta}_c)_n$ vanishes outside Ω_n. Then by Lemma 4.4.6 and the maximum principle

$$\begin{aligned}T\tilde{\zeta}_c(\mathbf{x}) - \frac{1}{2\pi}\log\left(\frac{c}{2a}\right)||\tilde{\zeta}_c||_1 &\leq \sum_{|n|>1} T_0(\tilde{\zeta}_c)_n + \frac{1}{2\pi}\int_{\Omega_0}\log\left(\frac{2a|\mathbf{x}-\bar{\mathbf{y}}|}{c|\mathbf{x}-\mathbf{y}|}\right)\tilde{\zeta}_c(\mathbf{y})d\mathbf{y} \\ &\quad + \frac{1}{2\pi}\int_{\Omega_0}\log\left(\frac{|\mathbf{x}-\bar{\mathbf{y}}|}{|\mathbf{x}-\mathbf{y}|}\right)((\tilde{\zeta}_c)_1(\mathbf{y}) + (\tilde{\zeta}_c)_{-1}(\mathbf{y}))d\mathbf{y}.\end{aligned}$$

If $|n| > 1$ then for $\mathbf{x} \in \Omega_0$ and $\mathbf{y} \in \Omega_n$, $|\mathbf{x} - \mathbf{y}| \geq (|n| - 1)t$ and

$$\begin{aligned}\sum_{|n|>1} T_0(\tilde{\zeta}_c)_n(\mathbf{x}) &= \sum_{|n|>1}\frac{1}{4\pi}\int_{\Omega_n}\log\left(1 + \frac{4x_2y_2}{|\mathbf{x}-\mathbf{y}|^2}\right)(\tilde{\zeta}_c)_n(\mathbf{y})d\mathbf{y} \\ &\leq \frac{b^2||\omega_0||_1}{\pi t^2}\sum_{|n|>1}\frac{1}{(|n|-1)^2} \\ &\leq \frac{\alpha_1^2||\omega_0||_1}{\pi}\sum_{|n|>1}\frac{1}{(|n|-1)^2}. \qquad (4.5.25)\end{aligned}$$

It is shown in Appendix A that there exist positive constants $A_1, A_2, A_3 > 0$ depending only on a and p such that

$$\frac{1}{2\pi}\int_B \log\left(\frac{2a|\mathbf{x}-\bar{\mathbf{y}}|}{c|\mathbf{x}-\mathbf{y}|}\right)\tilde{\zeta}_c(\mathbf{y})d\mathbf{y} \leq \begin{cases} (A_1 + A_2|\log x_2|)||\omega_0||_p & \text{if } x_2 \geq a, \\ A_3||\omega_0||_p & \text{if } 0 < x_2 \leq a.\end{cases} \qquad (4.5.26)$$

Note that for $\mathbf{y} \in \Omega_0 \backslash B$ we have $|\mathbf{x}-\mathbf{y}| \geq Ra/c$ and $|\mathbf{x}-\bar{\mathbf{y}}| \leq (2b+t)$, hence

$$\frac{1}{2\pi}\int_{\Omega_0\backslash B} \log\left(\frac{2a|\mathbf{x}-\bar{\mathbf{y}}|}{c|\mathbf{x}-\mathbf{y}|}\right)\tilde{\zeta}_c(\mathbf{y})d\mathbf{y} \leq \frac{1}{2\pi}\log\left(\frac{2(2b+t)}{R}\right)\int_{\Omega_0\backslash B}\tilde{\zeta}_c(\mathbf{y})d\mathbf{y}. \quad (4.5.27)$$

If $|x_1| < t/4$ then for all $\mathbf{y} \in \Omega_1 \cup \Omega_{-1}$ we have $|\mathbf{x}-\bar{\mathbf{y}}| \leq 2(b+t)$ and $|\mathbf{x}-\mathbf{y}| > t/4$. Hence

$$\frac{1}{2\pi}\int_{\Omega_0} \log\left(\frac{|\mathbf{x}-\bar{\mathbf{y}}|}{|\mathbf{x}-\mathbf{y}|}\right)((\tilde{\zeta}_c)_1(\mathbf{y})+(\tilde{\zeta}_c)_{-1}(\mathbf{y}))d\mathbf{y} \leq \frac{||\omega_0||_1}{\pi}\log\frac{8(b+t)}{t} \leq \frac{||\omega_0||_1}{\pi}\log(8(\alpha_1+1)). \quad (4.5.28)$$

Applying the estimates (4.5.25), (4.5.26), (4.5.27) and (4.5.28) and rearranging (4.5.24) we obtain

$$\frac{1}{2\pi}\log\left(\frac{R}{2(2b+t)}\right)\int_{\Omega_0\backslash B}\tilde{\zeta}_c(\mathbf{y})d\mathbf{y} \leq \begin{cases} (A_1+A_2|\log x_2|)||\omega_0||_p - x_2 + M & \text{if } x_2 \geq a, \\ A_3||\omega_0||_p - x_2 + M & \text{if } 0 < x_2 \leq a \end{cases} \quad (4.5.29)$$

hence

$$\log\left(\frac{R}{2(2b+t)}\right)\int_{\Omega_0\backslash B}\tilde{\zeta}_c(\mathbf{y})d\mathbf{y} < M. \quad (4.5.30)$$

Let $R = 2(2b+t)e^{2M/||\omega_0||_1}$. Then

$$\int_{\Omega_0\backslash B}\tilde{\zeta}_c(\mathbf{y})d\mathbf{y} < \frac{||\tilde{\zeta}_c||_1}{2} \quad (4.5.31)$$

for almost all $\mathbf{x} \in S_c$ with $|x_1| < t/4$.

Choose $C_4 \geq C_3$ depending on ω_0, α_1 and p only, such that $Ra/c < t/16$ for $c > C_4$. Fix $c > C_4$ and suppose ess diam$(S_c) > 4Ra/c$. Then $S_c = S_c' \cup N$ where diam$(S_c') > 4Ra/c$, $\mu_2(N) = 0$ and (4.5.31) holds for all $\mathbf{x} \in S_c'$ with $|x_1| < t/4$. We may also assume that for all $\mathbf{x} \in S_c'$, $T\tilde{\zeta}_c(\mathbf{x}) - x_2 > \gamma$ and, by Steiner-symmetry of $T\tilde{\zeta}_c$, that $T\tilde{\zeta}_c(\cdot, x_2)$ is a symmetric decreasing function.

By (4.5.15) and the fact that diam$(S_c') > 4Ra/c$ there exist $\mathbf{x}' = (x_1', x_2') \in S_c'$ and $\mathbf{x}'' = (x_1'', x_2'') \in S_c'$ with $|\mathbf{x}'-\mathbf{x}''| > 2Ra/c$ and $|x_1'| < t/16$. Furthermore, since $T\tilde{\zeta}_c(\cdot, x_2'')$ is a symmetric decreasing function, there exists $\mathbf{x}''' = (x_1''', x_2''') \in S_c'$

with $|\mathbf{x}''' - \mathbf{x}'| > 2Ra/c$ and $|x_1'''| < t/4$. Then $B_{Ra/c}(\mathbf{x}') \cap B_{Ra/c}(\mathbf{x}''') = \emptyset$ and

$$||\tilde{\zeta}_c||_1 \leq \int_{\Omega_0 \backslash B_{Ra/c}(\mathbf{x}')} \tilde{\zeta}_c + \int_{\Omega_0 \backslash B_{Ra/c}(\mathbf{x}''')} \tilde{\zeta}_c < \frac{||\tilde{\zeta}_c||_1}{2} + \frac{||\tilde{\zeta}_c||_1}{2} = ||\tilde{\zeta}_c||_1$$

which is a contradiction. Hence ess diam$(S_c) \leq 4Ra/c$. □

By Lemma 4.5.2 there exists $C(\omega_0, p, \alpha_1)$ such that ess diam$(S_c) \leq t/4$ for $c > C(\omega_0, p, \alpha_1)$. Hence the support of $\tilde{\zeta}_c$ is bounded away from Γ_0 if c is sufficiently large. It remains to show that the support of $\tilde{\zeta}_c$ also avoids $\partial\Omega_0 \backslash \Gamma_0$. From the lower bound for γ (4.5.23) it is clear that $\gamma > 0$ if c is sufficiently large. Hence $T\tilde{\zeta}_c - x_2 - \gamma$ is negative on $\partial\Omega_0 \backslash \Gamma_0$ and, since $S_c = \{T\tilde{\zeta}_c - x_2 > \gamma\}$ except for a set of measure zero, we obtain the following result.

THEOREM 4.5.3 *Let $2 < p < \infty$ and let $\omega_0 \in L^p(\mathbb{R}^2)$ be a non-zero non-negative function vanishing outside a set of measure πa^2. Let $b, t > \max\{4, 2a\}$ with $b/t \leq \alpha_1$. Let $0 < \lambda < 1$ and let $\tilde{\omega}$ be a Steiner-symmetric maximiser of Ψ_λ relative to the set of rearrangements of ω_0 supported on Ω_0^λ. Then, letting $\psi = T_\lambda \tilde{\omega}$, we have*

$$-\Delta\psi = \tilde{\omega} = \phi \circ (\psi - \lambda x_2)$$

almost everywhere in Ω_0^λ for some increasing function ϕ. Furthermore, there exists $\lambda_0 = \lambda_0(\omega_0, p, \alpha_1)$ such that for $0 < \lambda < \lambda_0$ the support of $\tilde{\omega}$ is bounded away from $\partial\Omega_0^\lambda$.

The extension of $\psi - \lambda x_2$ to a function on Ω^λ that is t/λ-periodic in x_1 provides the stream function for a steady flow of an ideal fluid in Ω^λ that contains patches of vorticity. The vorticity in each patch is a rearrangement of ω_0.

Appendix A

Let $0 < a < \infty$, $2 < p < \infty$, $R \geq 1$ and $c \geq 2a$. Let $v \in L^p(\Pi)$ be a non-negative function vanishing outside a set of area πa^2 and define $v_c(\mathbf{x}) = c^2 v(c\mathbf{x})$. For $\mathbf{x} \in \Pi$ define

$$I(\mathbf{x}) = \int_{B_{Ra/c}(\mathbf{x})} \log\left(\frac{2a|\mathbf{x}-\bar{\mathbf{y}}|}{c|\mathbf{x}-\mathbf{y}|}\right) v_c(\mathbf{y})d\mathbf{y}.$$

Then there are positive constants M_1, M_2 and M_3 such that

$$\begin{aligned} I(\mathbf{x}) &\leq (M_1 + M_2|\log x_2|)||v||_p \quad \text{if } x_2 \geq a, && \text{(A.0.1)} \\ I(\mathbf{x}) &\leq M_3||v||_p \qquad\qquad\qquad \text{if } x_2 \leq a. && \text{(A.0.2)} \end{aligned}$$

Proof For $\mathbf{x}, \mathbf{y} \in \Pi$ let $\rho = |\mathbf{x}-\mathbf{y}|$, $\delta = |\mathbf{x}-\bar{\mathbf{y}}|$. Then

$$\delta^2 = \rho^2 + 4x_2y_2.$$

If $\rho \geq x_2/c$ then

$$\delta \leq |\mathbf{x}-\bar{\mathbf{x}}| + |\bar{\mathbf{x}}-\bar{\mathbf{y}}| = 2x_2 + \rho \leq (2c+1)\rho.$$

Hence

$$\begin{aligned} \int_{\rho \geq x_2/c} \log\left(\frac{2a|\mathbf{x}-\bar{\mathbf{y}}|}{c|\mathbf{x}-\mathbf{y}|}\right) v_c(\mathbf{y})d\mathbf{y} &\leq \log\left(\frac{2a(2c+1)}{c}\right)||v||_1 \\ &\leq \log(4a+1)||v||_1. && \text{(A.0.3)} \end{aligned}$$

Let $\hat{v}_c$ denote the rearrangement of v_c as a decreasing function of ρ only on

$\mathbb{R}^2$. If $\rho \le x_2/c$ then $\delta \le (2c+1)x_2/c$ and if $x_2 \ge a$

$$\begin{aligned}
\int_{\rho\le x_2/c} \log\left(\frac{2a|\mathbf{x}-\bar{\mathbf{y}}|}{c|\mathbf{x}-\mathbf{y}|}\right) v_c(\mathbf{y})d\mathbf{y} &\le \int_{\rho\le x_2/c} \log\left(\frac{2a(2c+1)x_2}{c^2|\mathbf{x}-\mathbf{y}|}\right) v_c(\mathbf{y})d\mathbf{y} \\
&\le \int_{\rho\le a/c} \log\left(\frac{2a(2c+1)x_2}{c^2|\mathbf{x}-\mathbf{y}|}\right) \hat{v}_c(\mathbf{y})d\mathbf{y} \\
&= \int_{\tilde{\rho}\le a} \log\left(\frac{2a(2c+1)x_2}{c\tilde{\rho}}\right) \hat{v}(\mathbf{u})d\mathbf{u}
\end{aligned}$$

where $\tilde{\rho} = |c\mathbf{x}-\mathbf{u}|$ and $\hat{v}$ is the rearrangement of v as a decreasing function of $\tilde{\rho}$ only.

Hence

$$\begin{aligned}
&\int_{\rho\le x_2/c} \log\left(\frac{2a|\mathbf{x}-\bar{\mathbf{y}}|}{c|\mathbf{x}-\mathbf{y}|}\right) v_c(\mathbf{y})d\mathbf{y} \\
\le\ & \left(2\pi \int_0^a \left|\log\left(\frac{c\tilde{\rho}}{2a(2c+1)}\right)\right|^q \tilde{\rho}d\tilde{\rho}\right)^{1/q} ||\hat{v}||_p + |\log x_2|||\hat{v}||_1.
\end{aligned}$$

Substituting $z = c\tilde{\rho}/2a(2c+1)$

$$\begin{aligned}
&\int_{\rho\le x_2/c} \log\left(\frac{2a|\mathbf{x}-\bar{\mathbf{y}}|}{c|\mathbf{x}-\mathbf{y}|}\right) v_c(\mathbf{y})d\mathbf{y} \\
\le\ & \left(2\pi \int_0^{c/2(2c+1)} \left(\frac{2a(2c+1)}{c}\right)^2 |\log z|^q z dz\right)^{1/q} ||v||_p + |\log x_2|||v||_1 \\
\le\ & (4a+1)^{2/q} \left(2\pi \int_0^1 z|\log z|^q dz\right)^{1/q} ||v||_p + (\pi a^2)^{1/q}|\log x_2|||\hat{v}||_p \\
\le\ & (M_1 + M_2|\log x_2|)||v||_p \qquad \text{(A.0.4)}
\end{aligned}$$

for some positive constants M_1, M_2 since $\int_0^1 z|\log z|^q dz$ is finite. Combining (A.0.3) and (A.0.4) we obtain (A.0.1).

If $0 \le x_2 \le a$ then

$$\begin{aligned}
\int_{\rho\le x_2/c} \log\left(\frac{2a|\mathbf{x}-\bar{\mathbf{y}}|}{c|\mathbf{x}-\mathbf{y}|}\right) v_c(\mathbf{y})d\mathbf{y} &\le \int_{\rho\le x_2/c} \log\left(\frac{2a(2x_2+x_2/c)}{c\rho}\right) v_c(\mathbf{y})d\mathbf{y} \\
&\le \left(2\pi \int_0^{x_2/c} \left|\log\left(\frac{c\rho}{2ax_2(2+1/c)}\right)\right|^q \rho d\rho\right)^{1/q} ||v_c||_p.
\end{aligned}$$

Substituting $z = c\rho/(2ax_2(2+1/c))$ and noting that $||v_c||_p = c^{2/q}||v||_p$ we have, for $0 \leq x_2 \leq a$,

$$
\begin{aligned}
& \int_{\rho \leq x_2/c} \log\left(\frac{2a|\mathbf{x}-\bar{\mathbf{y}}|}{c|\mathbf{x}-\mathbf{y}|}\right) v_c(\mathbf{y})d\mathbf{y} \\
\leq \; & \left(2\pi \int_0^{1/2a(2+1/c)} z|\log z|^q dz\right)^{1/q} \left(\frac{2ax_2(2+1/c)}{c}\right)^{2/q} c^{2/q}||v||_p \\
\leq \; & \left(2\pi \int_0^{1/4a} z|\log z|^q dz\right)^{1/q} ((4a+1)x_2)^{2/q}||v||_p \\
\leq \; & M_3 x_2^{2/q}||v||_p
\end{aligned}
\tag{A.0.5}
$$

for some positive constant M_3.

Combining (A.0.3) and (A.0.5) we obtain (A.0.2). □

Appendix B

Let $U \subset \mathbb{R}^2$ be a bounded domain. For $g \in L^p(U)$, $2 < p < \infty$, define

$$Kg(\mathbf{x}) = \int_U \frac{1}{2\pi} \log \frac{1}{|\mathbf{x}-\mathbf{y}|} g(\mathbf{y}) d\mathbf{y}.$$

Then $K : L^p(U) \to W^{2,p}(U)$ is bounded and $-\Delta Kg = g$.

Proof Let $r = \mathrm{diam}(U)$ and suppose that $r > 1$. Then

$$Kg(\mathbf{x}) = \int_{|\mathbf{x}-\mathbf{y}|>1} \frac{1}{2\pi} \log \frac{1}{|\mathbf{x}-\mathbf{y}|} g(\mathbf{y}) d\mathbf{y} + \int_{|\mathbf{x}-\mathbf{y}|<1} \frac{1}{2\pi} \log \frac{1}{|\mathbf{x}-\mathbf{y}|} g(\mathbf{y}) d\mathbf{y}$$

and, letting q denote the conjugate exponent of p,

$$\begin{aligned} |Kg(\mathbf{x})| &\leq \int_U \frac{1}{2\pi} |\log r| |g(\mathbf{y})| d\mathbf{y} + \left((2\pi)^{1-q} \int_0^1 |\log \rho|^q \rho d\rho \right)^{1/q} ||g||_p \\ &\leq M_1(r) ||g||_p \end{aligned}$$

since $\int_0^1 |\log \rho|^q \rho d\rho = \int_{-\infty}^0 |z|^q e^{2z} dz < \infty$. Also

$$\begin{aligned} |\nabla Kg(\mathbf{x})| &\leq \int_U \frac{1}{2\pi|\mathbf{x}-\mathbf{y}|} g(\mathbf{y}) d\mathbf{y} \\ &= \int_{|\mathbf{x}-\mathbf{y}|>1} \frac{1}{2\pi|\mathbf{x}-\mathbf{y}|} g(\mathbf{y}) d\mathbf{y} + \int_{|\mathbf{x}-\mathbf{y}|<1} \frac{1}{2\pi|\mathbf{x}-\mathbf{y}|} g(\mathbf{y}) d\mathbf{y} \\ &\leq \frac{1}{2\pi} ||g||_1 + \left((2\pi)^{1-q} \int_0^1 \rho^{1-q} d\rho \right)^{1/q} ||g||_p \\ &\leq M_2(r) ||g||_p \end{aligned}$$

since $\int_0^1 \rho^{1-q} d\rho$ is finite. The result follows from [23, Theorem 9.9]. □

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